

UNIVERSITE DE GENEVE

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On the critical behaviour of models including third order couplings.

(Sur le comportement critique de modèles avec couplage de
troisième ordre)

THESE

présentée à la faculté des sciences
de l'Université de Genève
pour obtenir le grade de
docteur ès sciences, mention physique

par

Andreas MALASPINAS

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Athènes

Thèse N° 1773

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La faculté des sciences, sur le préavis de Messieurs C.P. ENZ, professeur ordinaire et directeur de thèse, L. THOMAS, professeur (Université de Bâle) et N. SZABO, chargé de recherches (Physique théorique) autorise l'impression de la présente thèse, sans exprimer d'opinion sur les propositions qui y sont énoncées.

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RESUME

Le but de ce travail est de discuter le comportement critique des modèles avec couplage de troisième ordre.

Nous utilisons la théorie du groupe de renormalisation développée par K.G. Wilson, (Wi 11) et (Wi 12), pour étudier les phénomènes critiques et les transitions de phases. Etant donné que cette théorie est, dans un certain sens, une extension de la théorie du champ moléculaire (La Li), nous comparons aussi les prédictions de ces deux théories. Du point de vue quantitatif, il est bien connu que les deux théories donnent en général des résultats différents. Par contre, leurs prédictions qualitatives (ordre de la transition) coïncident dans la plupart des cas.

Dans la première partie, nous discutons des modèles où le paramètre de couplage du troisième ordre est constant. Si le modèle est choisi symétrique par rotation sans axe distingué, alors le terme de troisième ordre inhibe un comportement continu en accord avec la théorie du champ moléculaire. Par contre, un modèle avec un axe distingué et une constante de couplage du troisième ordre différente de zéro peut exhiber une transition de phase continue. Bien que l'aspect quantitatif (exposants critiques) ne diffère pas des modèles sans terme de troisième ordre, on trouve une grande variété de régions critiques possibles (plusieurs points fixes) dont certaines particulières au modèle discuté. Que ce genre de modèle puisse subir une transition continue est en accord avec la théorie du champ moléculaire. Mais l'approche du groupe de renormalisation permet d'affiner l'analyse et de clarifier les conditions sous lesquelles cette transition a lieu.

Dans la deuxième partie, nous examinons un modèle où le paramètre de couplage de troisième ordre dépend linéairement du vecteur d'ondes. Dans ce cas, la comparaison avec la théorie de Landau n'est plus possible puisque dans cette théorie les variations spatiales sont négligées. Le comportement critique de ce modèle (qui n'a pas de symétrie) s'avère surprenant. En particulier, la dimension anormale η est proportionnelle, dans certains cas, à ϵ ($\epsilon = 4 - d$, $d < 4$ la dimension de l'espace) plutôt qu'à ϵ^2 . Ceci pose peut-être un problème en ce qui concerne la formulation de l'hypothèse de l'universalité. Enfin, il serait indiqué de discuter en détail un modèle similaire mais symétrique par rotation.

PREFACE

It is usual, in the beginning of a doctoral thesis, to thank a certain number of people involved directly or indirectly in the elaboration of the work presented. Before sacrificing to this custom let me state that no acknowledgment should be taken as a guarantee of the quality of the research following. For this, for its drawbacks or its merits, the whole responsibility is taken by the author.

Having stated this it is with even more pleasure that I thank Charles P. Enz for accepting me in his research group and hence for being my thesis director, for his participation in the jury and, last but not least, for the human quality of his contact.

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CHAPTER 0

INTRODUCTION

§01 Brief historical survey

The problem of phase transitions (P.T) was first given a satisfying theoretical description, in the case of the liquid-gas transition, by V. der Waals in 1873. He discovered the equation of state

$$01.1 \quad \left(p + \frac{a}{V^2}\right) (V - b) = RT$$

with p being the pressure, V the volume, T the temperature and a and b material constants describing the internal forces of the system. This equation not only corrected quantitatively the ideal gas law

$$pV = RT$$

but allowed for a critical point (P_c , V_c , T_c) which gave a good description of the liquid-gas critical point.

In fact, in 1869, T. Andrews (An) published his measurements of the P - V diagrams of "carbonic acid" which were amazingly accurate.

In this publication he added the following comment: "In the foregoing observations I have avoided all reference to the molecular forces brought into play in these experiments. The resistance of liquids and gases to external pressure tending to produce a diminution of volume, proves the existence of an internal force of an expansive or resisting character. On the other hand, the sudden diminution of volume, without application of additional pressure externally, which occurs when a gas is compressed, at any temperature below the critical point, to the point at which liquefaction begins,

can scarcely be explained without assuming that a molecular force of great attractive power comes into operation and overcomes the resistance to diminution of volume, which commonly requires the application of external force..."

This is a perfect description of Ol.1.

One of the most interesting features of Ol.1 is that one can eliminate the material constants a and b by a change of variables. The equation obtained (LaLi) reads

$$\text{Ol.2} \quad \left(\rho' + \frac{3}{V'} \right) (3V' - 1) = 8T'$$

This matter of fact is known as the "law of corresponding states".

In the beginning of the 20th century physicists attacked the problem of ferromagnets. The corresponding theoretical work was done essentially by Pierre Weiss (We). He assumed that the individual magnetic moments interact through an effective "molecular field" proportional to the average magnetisation. This mean (or molecular) field theory gave good qualitative results, but failed quantitatively, like the V.d.Waals theory.

Along with the interest in magnets appeared the first exact models of Ising (Is) and Heisenberg (He). Although the one dimensional model of Ising showed no critical behaviour its higher dimensional extensions play a very important role in the theory of phase transitions.

The mean field theories are unified in what is called the "Landau theory" (La) which will be sketched later.

The next breakthrough in the understanding of critical phenomena is due to Domb (Do) Widom (Wi) and most of all Leo Kadanoff (Ka1). He formulated the scaling idea and stressed the importance of correlation length.

Although Kadanoff himself (Ka2) devised a mathematical procedure which allowed for the calculation of the relevant quantities in the critical region (i.e. the critical exponents), it was K.Wilson (Wi1), (Wi2) who, by reformulating Kadanoffs ideas and using the renormalisation group (R.G.) approach of field theory (GeLo), (StPe) made the critical exponents accessible to calculation. The R.G. approach to critical phenomena was studied about the same time by Jona-Lasinio and Di Castro (CaLa) and a little later by Brezin, Zinn-Justin and Le Guillou (BrJuGu). It will be briefly reviewed later in this introduction since it underlies all this work.

§02. Some definitions

Since, in what follows, we talk often about phase transitions it is probably usefull to define what we mean.

For this we can give a kind of classification of P.T. based on the notion of symmetry change. So we consider a system which has some symmetry and we change continually its temperature until we reach a critical temperature T_c where the following things can happen.

1. Although different quantities observed show a peculiar behaviour as functions of temperature, the symmetry of the system doesn't change.

Examples : Liquid-gas transition
 Metal-Insulator transition

2. The symmetry groups are different for $T > T_c$ and $T < T_c$, but not in a group-subgroup relation.

Example : transitions under polymorphic varieties
 of solids

3. The symmetry group of the less symmetric ($T < T_c$) phase is a subgroup of the symmetry group of the most symmetric one ($T > T_c$).

Example : ferromagnets

We shall look at the cases where we can define an order parameter, that is a characteristic quantity which is zero for $T > T_c$ and different from zero at $T < T_c$. We will ask that the order parameter changes continuously at $T = T_c$. And we will call such a transition a continuous one. An other definition, which bypasses the problem of identifying an order parameter is to define a continuous P.T. by saying that in such a transition the correlation length diverges at $T = T_c$.

As a matter of fact the two definitions are equivalent if we can define an order parameter and this is immediate for the case 3. above, the order parameter being defined, in general, as a vector that describes the symmetry change. In the case 1. it is sometimes possible to find one like $\Delta \rho = \rho_L - \rho_c$ the difference between the liquid density and the critical density.

In a continuous phase transition different observable quantities obey a power law near the critical temperature. Explicitly, if we set

$$\Delta T = \frac{T - T_c}{T_c}$$

and we consider some quantity $g(\Delta T)$ assumed to be positive and continuous for ΔT , positive and small, we define a critical point exponent by the limit

$$\lambda = \lim_{\Delta T \rightarrow 0^+} \frac{\log[g(\Delta T)]}{\log(\Delta T)}$$

It follows that near T_c

$$g(\Delta T) \sim (\Delta T)^\lambda$$

There are many critical exponents. For example the ones corresponding to the behaviour of the order parameter (β), the specific heat (α), the correlation length (ν) etc. For a complete treatment the book of E. Stanley (St) can be consulted.

Since in what follows we will focus on the critical exponent ν and on the anomalous dimension η we will define the latter. At $T = T_c$ we may look at the pair correlation function $G(r)$ at zero external field. The behaviour observed is

$$G(r) \sim \frac{1}{r^{d-2+\eta}}$$

In fact mean field theory or the Ornstein-Zernike theory predict $\eta = 0$ (OrZe). The exponent η was introduced by Fisher (Fi) and is well established since.

Measurements and exactly soluble models gave rise to the universality hypothesis which states (Ka2) that the critical exponents depend on

1. The dimensionality of the system
2. The degrees of freedom of the order parameter
3. Maybe some other unknown things.

What is more universal than the critical exponents themselves, are the exponent inequalities which relate them. They are many like the famous Rushbrooke inequality (Ru)

$$\alpha + 2\beta + \gamma \geq 2$$

the Griffiths inequalities, the Fisher inequalities etc. (St), (Gr).

Scaling theory predicts equalities instead of inequalities and this is well confirmed by experimental data. It follows that one needs two critical point exponents to establish the rest of them. They are, traditionally, ν and η .

§03. Sketch of the Landau theory.

In order to use a uniform notation and language with the sketch of the R.G. method, which will follow, we will adopt a functional integral formalism. A detailed study of functional derivatives and integrals can be found in (Ta).

We will suppose that our system is described by a probability distribution.

$$03.1 \quad P = e^{-F\{\Phi\}}$$

with F being some functional of some random variable $\Phi(\vec{x})$

$$03.2 \quad F\{\Phi\} = \int d\vec{x} f(\Phi(\vec{x}))$$

Information about the system is then given by the functional integral (the partition function)

$$03.3 \quad Z = \int \mathcal{D}\{\Phi(\vec{x})\} e^{-F\{\Phi\}}$$

This is the usual partition function in the following sense :

Assume that a system is described by some order parameter $\Phi(\vec{x})$. Instead of writing

$$e^{-\beta F} = Z = \text{Tr} e^{-\beta H}$$

H being the Hamiltonian, $\beta = 1/kT$ and F the free energy, we first sum over all microscopic configurations corresponding to some $\phi(\vec{x})$ defining some probability distribution

$$e^{-F\{\phi\}} = \int d\vec{x} f(\phi(\vec{x}))$$

and we identify f with a local free energy divided by β . Then we sum over all possible $\phi(\vec{x})$, that is we get the path integral for Z . Actually the order parameter doesn't have to be a scalar. We take it to be so for simplicity reasons.

In the Landau theory we assume that we can expand f in powers of ϕ and this near the critical point. As an example take

$$f(\phi(\vec{x})) = \frac{1}{2} m \phi^2(\vec{x}) + u_4 \phi^4(\vec{x}) + \frac{1}{2} \sum_{i=1}^d c \left(\frac{\partial}{\partial x_i} \phi(\vec{x}) \right)^2$$

is called the Landau-Ginzburg form where higher order terms are neglected.

We then extremalise the exponential i.e. we solve the equation:

$$\frac{\delta F}{\delta \phi(\vec{x})} = m \phi(\vec{x}) + 4 u_4 \phi^3(\vec{x}) - c \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2} \phi(\vec{x}) = 0$$

We can now look for a homogeneous $\phi(\vec{x}) = \phi_0$ satisfying this equation. It follows

$$m \phi_0 + 4 u_4 \phi_0^3 = 0$$

The stability condition implies $u_4 > 0$ and we see that

$$\phi_0 = 0 \quad \text{for} \quad m \geq 0$$

$$\phi_0 = \left(-\frac{m}{4u_4} \right)^{1/2} \quad \text{for} \quad m < 0$$

Setting $m \sim T - T_c$ we find the critical exponent β to be equal to $1/2$. We can also calculate γ which turns out to be equal one.

If we don't look for a homogeneous order parameter but we drop u_4 we find, in Fourier space

$$F = \frac{1}{2} \sum_{\vec{q}} \phi^2(\vec{q}) (m + e q^2)$$

$$\langle \phi^2(\vec{q}) \rangle = \frac{\int d\{\phi(\vec{q})\} \phi^2(\vec{q}) e^{-F}}{\int d\{\phi(\vec{q})\} e^{-F}}$$

$$= \frac{1}{m + e q^2}$$

by the fluctuation dissipation theorem (En) the susceptibility behaves like

$$\chi \sim \frac{1}{m} \quad (\text{i.e. } \gamma = 1)$$

and by defining the correlation length

$$\xi^2 = \frac{e}{m} \quad (\text{i.e. } \nu = \frac{1}{2})$$

we find

$$G(\vec{q}) = \langle \phi^2(\vec{q}) \rangle = \frac{1}{(m (1 + \xi^2 q^2))}$$

For $\vec{r} = \vec{0}$ ($\vec{r} \rightarrow 0$)

$$G(r) \sim \frac{1}{r^{d-2}} e^{-r/\xi}$$

and we find the Ornstein-Zernike theory (OrZe) ($\eta = 0$) which is absurd for $d = 2$ giving a logarithmic divergence for $\xi(r)$.

The Landau theory is discussed in (La) (LaLi) for example, and the functional integral formulation in (Si), (Fe) and (AmZa).

§ 04. The role of dimensionality and the interaction range.

It is known that the critical exponents become classical (Landau) for $d = 4$. To explain this we give some version of the Ginzburg argument (Gi), (To).

We can define the interaction range by stating that the interaction has the form

$$V = \frac{1}{r^{d+\sigma}}$$

For $\sigma > 2$ we speak of short range interactions. For $0 < \sigma < 2$ we speak of long range interactions. Define, further, p to be the order of expansion of the free energy as

$$f_9 = m \phi^2 + u_p \phi^p + q^\sigma \phi^2$$

In Landau theory we have

$$\beta = \frac{1}{p-2}, \quad \nu = \frac{1}{\sigma}, \quad \sigma \leq 2$$

Intuitively $u_p \phi^p \xi^d \sim 1$ is dimensionless
 it follows that $u_p \sim (\Delta T)^{d\nu - p\beta}$
 and since $d\nu - p\beta = d/\sigma - p/(p-2)$ the Landau
 theory is correct if $d/\sigma \geq p/(p-2)$ since in this
 case $u_p \rightarrow 0$

We define the critical dimension to be

$$d_c = \frac{p\sigma}{p-2}$$

and we see that for $p = 4$ and $\sigma = 2$ $d_c = 4$ or better $d_c = 4$ for short range forces.

In the case $u_4 = 0$ and $u_6 \neq 0$ (tricritical point) we see that

$$d_c = 3$$

The problem of such an argument is that it uses Landau theory to give its limitations. That the classical exponents are correct for $d = 4$ follows from the Josephson inequality (Jo)

$$d\nu \geq 2 - \alpha \quad \text{which gives } d \geq 4 \text{ if we set } \nu = 1/2 \text{ and } \alpha = 0$$

§ 05. Brief sketch of the R.G. method.

There are many ways of presenting the R.G. ideas. We will follow the presentation of S. Ma (Ma) although the standard reference are (WiKo) or (BrJuGu).

This sketch has to be very concise and it certainly cannot replace any of the references given. Its aim is just to give some autonomy to what follows.

We will start again from the probability distribution

$$05.1 \quad \rho = \exp(-F\{\phi\}) = \exp(-\int P_L(\phi) d\vec{x})$$

where we replaced the f of § 03 by a P_L to indicate that we have to do with an L -order polynomial in the random variables (fields).

Now, knowledge and intuition about critical phenomena tell us that in the critical region only long distances play a role (or small wave vectors), because the system is controlled by large fluctuations of the order parameter and the correlation length, ξ , between these fluctuations becomes the characteristic length of the system. But ξ diverges like $(T - T_c)^{-\nu}$, $\nu > 0$, that is it becomes very large near T_c . We assume then that we can integrate out large momenta and write, in Fourier space.

$$\rho = \exp(-F\{\phi\}) = \exp(-\int \hat{P}_L(\phi))$$

where we use, abusively, the same symbols F and P_L . Explicitly

$$05.2 \quad P_L(\phi) = \sum_{k=2}^L \sum_{i_1, \dots, i_k=1}^N u_{i_1, \dots, i_k} \phi_{i_1}(\vec{q}_1) \dots \phi_{i_k}(\vec{q}_k) \delta(\sum_{i=1}^k \vec{q}_i)$$

where Λ is some cut-off, N is the number of components and $k \geq 2$ since we don't want external fields. Of course there are restrictions on the u 's, to begin with the last non vanishing u must correspond to a pair term and must be positive to ensure integrability of the probability distribution. Other restrictions are imposed according to the symmetries of the

system under consideration. Luckily it turns out that only a small number of u 's are needed to discuss critical behaviour.

Based on Kadanoff's (Ka) ideas Wilson (Will) devised a transformation which consist in integrating out large momenta and then rescale the fields and the momenta in such a way that the cut-off Λ remains the same.

This transformation can be written as

$$05.3 \quad \rho' = \left[\int \mathcal{D}\{\phi_s\} e^{-F\{\phi\}} \right] \phi(q) \rightarrow \int_s \phi(q')$$

where ϕ_s means the random variables restricted on $1/s < q < 1$, $s \geq 1$ and $\bar{q}' = s \bar{q}$. It is clear that P' is equivalent to P as far as the random variables ϕ_q are concerned, ϕ_q meaning the ϕ restricted in $1/s < q$.

From 05.1, 05.2 it is clear that a set of u 's defines a probability distribution. If we write

$$05.4 \quad \mu = (\{u_{i_1 i_2}\} \dots \{u_{i_1 \dots i_k}\})$$

the transformation defined by 05.3 can be thought as a transformation on μ and we write

$$05.5 \quad \mu' = R_s \mu$$

From 05.3 it follows that if we ask

$$05.6 \quad J_s J_{s'} = J_{ss'}$$

then

$$05.7 \quad R_s R_{s'} = R_{ss'}$$

This (semi) group property justifies the name renormalisation group transformation for R_s . We will assume that the transformation R_s has some fixed point μ^* that is

$$05.8 \quad R_s \mu^* = \mu^*$$

On the other hand equation 05.6 tells us that

$$05.9 \quad J_s = s^\gamma$$

and we expect that 05.8 is valid for some proper choice of γ . Since

$$\phi(\bar{q}) \rightarrow s^\gamma \phi(s\bar{q})$$

γ can be interpreted as the dimension of ϕ in units of length. We set

$$05.10 \quad \gamma = 1 - \frac{1}{2}\eta \quad , \quad J_s = s^{1-\eta/2}$$

and η will be the anomalous dimension. Assume now that in μ -space we choose some point near μ^*

$$\mu = \mu^* + \delta\mu$$

We can then linearise 05.8 and write

$$05.11 \quad \delta\mu' = L_s \delta\mu$$

Suppose now that we can diagonalise L_s and find its eigenvalues $\lambda_1(s), \lambda_2(s), \dots, \lambda_j(s), \dots$ ordered by $\lambda_1 \geq \lambda_2 \geq \dots$

and the eigenvectors $e_1, e_2, \dots$

Since

$$L_s L_{s'} e_j = L_{ss'} e_j$$

it follows

$$\lambda_j(s) \lambda_j(s') = \lambda_j(ss')$$

and so

$$\lambda_j(s) = s^{\gamma_j}$$

with

$$\gamma_1 \geq \gamma_2 \geq \dots$$

Then we can write

$$\delta\mu = \sum w_i e_i$$

and hence

$$\delta\mu' = \sum w_i s^{\gamma_i} e_i$$

We will now consider $y_1 > 0$, $y_j < 0$ for all $j \neq 1$
 For the case $y_1, y_2 > 0$ see (Ri) (RiWe) (WeRi).

Then

$$05.12 \quad \delta\mu' = w_1 s^{y_1} e_1 + \mathcal{O}(s^{y_2})$$

It follows that if we want the system to be "pushed" towards μ^* under repeated application of R_s (or $\lim_{s \rightarrow \infty} R_s$) we must choose a point near μ^* such that $w_1 = 0$.
 Then 05.12 tells us that

$$\delta\mu' \xrightarrow{s \rightarrow \infty} 0$$

$w_1 = 0$ defines the "critical surface" in μ -space and w_1 is called a "relevant" variable.

The whole point of this discussion is that we can argue that if the temperature is near the critical temperature then $\mu(T)$ is close to μ^*

Given a temperature T , $\mu(T)$ corresponds to a point in μ -space. We suppose that if we vary T , $\mu(T)$ describes a smooth trajectory and that the R.G. transformation doesn't change anything to this matter of fact.

We can then argue that for T near T_c

$$05.13 \quad w_1(T) = a(T - T_c) + b(T - T_c)^2 + \dots$$

and assume $a \neq 0$

Then by 05.12, 05.13

$$05.14 \quad \delta\mu'(T) = L_s \delta\mu(T) = a(T - T_c) s^{y_1} e_1 + \mathcal{O}(s^{y_2})$$

We then set

$$y_1 = 1/\nu$$

and we identify the correlation length

$$\xi = |w_1|^{-\nu} \sim |T - T_c|^{-\nu}$$

It remains then to justify the definition of η in 05.10 and then to give a scheme allowing to actually calculate ν and η , the other exponents following by exponents equalities (see §02).

Consider the two point function calculated with some probability distribution P

$$G(\vec{q}, \mu) = \langle \phi(\vec{q}) \phi(-\vec{q}) \rangle_p$$

Then 05.3 implies that

$$\langle \phi(\vec{q}) \phi(-\vec{q}) \rangle_p = \int_s^2 \langle \phi(s\vec{q}) \phi(-s\vec{q}) \rangle_p$$

that is, with 05.9, 05.10

$$\begin{aligned} G(\vec{q}, \mu) &= s^{2\gamma} G(s\vec{q}, R_s \mu) & \text{for } q < \gamma/s \\ &= s^{2-\eta} G(s\vec{q}, R_s \mu) \end{aligned}$$

Let us now place ourselves near μ^* . Then

$$05.15 \quad G(\vec{q}, \mu(T)) = s^{2-\eta} G[s\vec{q}, \mu^* + a(T-T_c) s^{4\nu} e_1 + \delta(s^{1/2})]$$

by 05.14, and for large s .

But we can choose $s \sim 1/q$. Then for $T = T_c$

$$05.16 \quad G(\vec{q}, \mu(T_c)) \sim q^{-2+\eta}$$

which is the equation defining η (Fi).

We end here the presentation of the R.G. ideas concluding that the general scheme is established essentially independently of any particular model. For the interested reader we note that a discussion of the subject in terms of probability theory and especially in terms of the central limit theorem can be found in (Ca).

We have seen by now that R.G. ideas lead to the discussion of the critical behaviour in terms of fixed points. So, given a model, we need a way of finding the fixed points. Note that 05.11 and 05.12 allows us to keep only a limited number of terms in 05.2 since (Ma) the smaller the y 's become the less the corresponding variables influence the critical behaviour. They are labelled as "irrelevant" in the sense that they don't influence the stability of μ^* .

The actual calculations are based on some perturbation scheme with expansion parameters like $1/N$ where N is the number of components of the random variables (fields) (Ab) or $\epsilon = d_c - d$

where d_{ϵ} is a critical dimension in the sense of § 04. Most of the times $d_{\epsilon} = 4$. We take $\epsilon > 0$ throughout this work.

In the last case one assumes that the coupling parameters are proportional to some power of ϵ and one defines R_s in terms of an expansion in ϵ , the fixed points and the exponents being calculated as power series in ϵ with the guess that the series are asymptotic. To give the perturbation scheme explicitly consider again 05.1 and take

$$05.17 \quad F\{\phi\} = F_0\{\phi\} + F_1\{\phi\}$$

with

$$05.18 \quad F_0\{\phi\} = \frac{1}{2} \int \sum_{i,j}^N G_{0ij}^{-1}(\vec{q}) \phi_i(\vec{q}) \phi_j(-\vec{q}) \frac{d\vec{q}}{(2\pi)^d}$$

so that u_2 in 05.4 is given by

$$u_{2ij} = \frac{1}{2} \delta_{ij} G_{0ii}^{-1}$$

Since $F_0\{\phi\}$ is quadratic we can write

$$F_0\{\phi\} = F_0\{\phi_<\} + F_0\{\phi_>\}$$

with $\phi_<$ and $\phi_>$ as before.

Let us then use 05.3

$$\rho' = e^{-F'} = \left[e^{-F_0\{\phi\}} \langle e^{-F_1\{\phi\}} \rangle_{0,}, \int d\phi_>\{ \phi\} e^{-F_0\{\phi_>\}} \right]_{\phi \rightarrow \int_<\phi}$$

where $\langle \rangle_{0,}$ means a kind of partial trace (over $\phi_>$) with the gaussian distribution $\exp(-F_0)$. The last term is just a constant and we find, using the linked cluster theorem (see for example En))

$$\langle e^{-F_1} \rangle_{0,} = \exp \langle (e^{-F_1} - 1) \rangle_{0,}^c$$

where $\langle \rangle^c$ means connected,

$$05.19 \quad F' = \left[F_0 - \langle (e^{-F_I} - 1) \rangle_0^c \right] \phi(\vec{q}) \rightarrow T_S \phi(\vec{q}')$$

which give us F' appart from an additive constant.
Through 05.19 we have then defined

$$\mu' = R_S \mu$$

$$05.20 \quad \mu' = (\{G_{0i}^{-1}\}, \dots, \{u_{i_1, \dots, i_L}\}, \dots)$$

to a given order in ϵ

Linearisation of R_S yields ν and a classification of variables in relevant and irrelevant. We repeat that the partial trace is taken over $\vec{p}$, so in a graph expansion (see app.A1) the internal lines have wave vectors restricted to $\Lambda_S < q < \Lambda$ while the external ones have wave vectors restricted to $q < \Lambda_S$.

If we consider the right hand side of 05.19 before rescaling the fields and the momenta we find it to be of the form

$$05.21 \quad \frac{1}{2} \int \frac{d\vec{q}}{(2\pi)^d} \sum_{i=1}^N (G_{0i}^{-1}(\vec{q}) + \Sigma_{is}(\vec{q})) \phi_i(\vec{q}) \phi_i(-\vec{q}) + \dots$$

where Σ_s is the self-energy (En).

When we then rescale we find

$$05.22 \quad G_{0i}^{-1}(\vec{q}) = s^{2-\eta} [G_{0i}^{-1}(\vec{q}/s) + \Sigma_s(\vec{q}/s)]$$

05.22 allows us to calculate the exponent η .

In fact assume that we reached the fixed point μ^* i.e. $R_S \mu^* = \mu^*$

Then

$$05.23 \quad G_0^{-1}(\vec{q}) = S^{2-\eta} [G_0^{-1}(\vec{q}/s) + \Sigma_s^*(\vec{q}/s)]$$

We can then expand both sides in q^2

$$G_0^{-1}(\vec{q}) = m^* + e_1^* q^2 + \dots$$

$$\Sigma_s^*(\vec{q}) = \Sigma_s^*(0) + q^2 \left(\frac{\partial \Sigma_s^*(q)}{\partial q^2} \right)_{q=0} + \dots$$

If we then fix $e_1^* = 1$ which can be done by a suitable choice of the units of q we find by comparison

$$05.24 \quad \eta = \log \left(1 + \left(\frac{\partial \Sigma_s^*(q)}{\partial q^2} \right)_{q=0} \right)$$

The condition $e_1^* = 1$ is one possible choice. There are situations where other choices are preferable (see § 11 and § 12).

In what follows the exponents will be calculated to lowest order in ϵ . This has the consequence to restrict very much the number of parameters involved in μ .

§ 06. Third-order couplings.

It is the object of the following chapters to use the techniques introduced in § 05 on models with third order terms.

In chapter I the coupling constants are taken to be wave vector independent in chapters II and III particular wave vector dependence of the third-order term is discussed.

The problem of third-order terms is well illustrated by the simple scalar model

$$06.1 \quad F\{\phi\} = \frac{1}{\kappa} \int d\vec{x} \left[m \phi^2(\vec{x}) + \sum_{i,j=1}^d \left(\frac{\partial}{\partial x_i} \phi(\vec{x}) \right)^2 \right] \\ + \int d\vec{x} u_3 \phi^3(\vec{x}) + \int d\vec{x} u_4 \phi^4(\vec{x})$$

Landau theory on such a model excludes any continuous transition as long as $u_3 \neq 0$

On the other hand Alexander and Amit (AlAm) study this model using the known results about the Ising model (WiKo).

06.2

$$F\{\phi\} = \frac{1}{2} \int d\vec{x} \left[v \phi^2(\vec{x}) + \sum_{i=1}^d \left(\frac{\partial}{\partial x_i} \phi(\vec{x}) \right)^2 \right] + \int d\vec{x} u_4 \phi^4(\vec{x})$$

and show that it is, qualitatively, well suited for the study of the liquid-gas transition. Their conclusion, among other things, is that Landau theory gives a correct qualitative description in case the system undergoes a symmetry change, is illustrated by §11. What actually remains unchanged is that this model yields the same exponents as the Ising model exponents which is still not established by measures on gases and approximations on the three dimensional Ising model.

CHAPTER I

ON MODELS WITH WAVE-VECTOR INDEPENDENT
THIRD ORDER COUPLING PARAMETER

§ 11 A symmetric case

Consider a general model defined by

$$\begin{aligned}
 F = & \int \frac{1}{2} \sum_{i=1}^N (m + e q^2) \phi_i(\vec{q}) \phi_i(-\vec{q}) \\
 & + \int \sum_{k, \ell, m=1}^N u_{3k\ell m} \phi_k(\vec{q}_1) \phi_\ell(\vec{q}_2) \phi_m(\vec{q}_3) \delta\left(\sum_{i=1}^3 \vec{q}_i\right) \\
 & + \int \sum_{k, \ell, m, n=1}^N u_{4k\ell mn} \phi_k(\vec{q}_1) \phi_\ell(\vec{q}_2) \phi_m(\vec{q}_3) \phi_n(\vec{q}_4) \delta\left(\sum_{i=1}^4 \vec{q}_i\right)
 \end{aligned}$$

11.1

We impose the following conditions on this model :

- a. All the tadpole diagrams, that is diagrams of the type



$$\longleftrightarrow \langle \phi_k \rangle = 0, \forall k$$

disappear. This implies some rotation symmetry with no invariant axis, in ϕ -space.

- b. We wish to keep the propagator component-independent. So we require

$$\langle \phi_i^2 \rangle = \langle \phi_j^2 \rangle \quad \forall i, j$$

11.2

In other words renormalisation doesn't introduce mass-splitting. This condition is compatible with a.

In order to discuss, qualitatively, the recursion relations we assume

$$\begin{aligned}
 \text{I1.3} \quad m &\sim \epsilon + \mathcal{O}(\epsilon^2) \\
 u_{3k\ell m} &\sim \epsilon^{1/2} + \mathcal{O}(\epsilon^{3/2}) \\
 u_{4k\ell mn} &\sim \epsilon + \mathcal{O}(\epsilon^2) \\
 &\vdots \\
 u_{2\eta-1} &\sim \epsilon^{\frac{2\eta-3}{2}} + \mathcal{O}(\epsilon^{\frac{2\eta-1}{2}}) \\
 u_{2\eta} &\sim \epsilon^{\eta-1} + \mathcal{O}(\epsilon^\eta)
 \end{aligned}$$

where $\epsilon = 4-d$, d being the dimension .

By proceeding as explained in § 05 we get the following type of equation for $u_{3k\ell m}$:

$$\text{I1.4} \quad u_{3k\ell m}^{\ell+1} = s^{1+\epsilon/2-3\eta/2} \left[u_{3k\ell m}^\ell + g(\{u_{2\eta+1}^\ell\}, \{u_{2\eta}^\ell\}, m) \right]$$

with η being of leading order u_4^2 . This last fact can be shown by just calculating  which is the first diagram contributing to η . This will be done at a later stage, see I2.2 and Appendix A3.

Suppose now that there exist fixed points and

$$u_{3\alpha\beta\gamma}^* = u_{\alpha\beta\gamma}^0 \epsilon^{1/2} + \mathcal{O}(\epsilon^{3/2}) \neq 0$$

and correspondingly

$$u_{2\eta+1}^* = u_{2\eta+1}^0 \epsilon^{\frac{2\eta-1}{2}} + \mathcal{O}(\epsilon^{\frac{2\eta+1}{2}})$$

$$\text{I1.5} \quad u_{2\eta}^* = u_{2\eta}^0 \epsilon^{\eta-1} + \mathcal{O}(\epsilon^\eta)$$

$$m^* = m^0 \epsilon + \mathcal{O}(\epsilon^2)$$

If we look back to I1.4 we notice that g which is a polynomial in the coupling constants starts with powers like $u_3^3, u_3 u_4$ etc. that is with $\epsilon^{3/2}$.

The fixed point equation can then be written as

$$\text{II.6} \quad \sum_p \epsilon^p a_p(s) + (s-1) u_{3\alpha\beta\gamma}^0 = 0, \quad p > 0$$

with the a_p depending on the u 's and m .

This relation is to be taken in the sense of a power series in ϵ being set equal to zero. Since s , the scaling parameter, is different from 1 the only possibility left is

$$\text{II.7} \quad u_{3\alpha\beta\gamma}^0 = 0$$

This means that

$$u_{3\alpha\beta\gamma}^* \sim \epsilon^{3/2} u_{3\alpha\beta\gamma}^1 + \mathcal{O}(\epsilon^{5/2})$$

We can now rewrite an equation like II.4 and repeat the same steps. This would lead to

$$u_{3\alpha\beta\gamma}^1 = 0$$

and so on.

It follows that $u_{3\alpha\beta\gamma}^* = 0$ to every order in ϵ . Together with u_3 all the u_{2n+1} are zero. This can be seen again by a step by step procedure. For example if

$$u_5^* = u_5^0 \epsilon^{3/2} + \mathcal{O}(\epsilon^{5/2})$$

the condition $u_3^0 = 0$ implies $u_5^0 = 0$ and so on.

What we have forgotten to talk about is that if we consider higher orders in ϵ , then we should take into account the wave vector dependence of the coupling constants. But this doesn't change anything to the qualitative discussion presented above. That is, the constant parts of the odd coupling parameters have to be zero at the fixed point. If $u_3^* = 0$ would correspond to a stable fixed point then we would have to do with a system that exhibits a continuous P.T. even if the initial u_3 is different from nought. But this is not so as can be seen from II.4 because of the factor s .

These conclusions seem to be related to the critical dimensions. In fact, according to § 05 we have fixed γ_s by requiring $e^* = 1$ (see II.1). Then we can calculate

$$\text{II.8} \quad u_{2n+1}^{p+1} \sim s^{(3-2n)d/2 + (2n-1) - \eta(2n-1)/2} (u_{2n-1}^p + \dots)$$

For $u_{2n-1}^* \neq 0$ we must require that

$$\text{I1.9} \quad d_c = \frac{4n-2}{2n-3}$$

for example $d_c = 6$ for $n = 2$ or $10/3$ for $n = 3$ etc.
But this doesn't help us because all the higher terms tend to zero making the system unstable i.e. the probability distribution non-integrable. As an example consider $d_c = 6$

Then since

$$\text{I1.10} \quad u_{2n}^{l+1} \sim s^{d(1-n)+2n-\eta n} (u_{2n}^l + \dots)$$

we see that for $d_c = 6$

$$u_{2n}^{l+1} \sim s^{-4n+6} (u_{2n}^l + \dots)$$

and so $u_{2n} \rightarrow 0$ for $n \geq 2$

The same is true for u_{2n-1} for $n > 2$

This situation should not be confused with the tri(poly)critical case (RiWe), (WeRi) where some u_{2n} is positive.

We can notice that I1.10 gives a bound for the applicability of the R.G. method since it implies

$$d_c = \frac{2n}{n-1} > 2 \quad \text{for} \quad n > 1$$

Another way of dealing with such models is by considering the recursion equation for m :

$$\text{I1.11} \quad m^{l+1} = \int^2 s^{-d} (m^l + (u_3^l)^2 A(\lambda, s) + u_4^l B(\lambda, s) + \dots)$$

where we drop the indices to simplify (for details see appendix A1).

If we choose $d_c = 4$ then $A(\lambda, s) \sim \log s$ and the choice of γ as in § 05 leads to

$$m^* \sim \log s$$

that is, m diverges or the correlation length goes to zero. The way out would be to change the definition of γ in such a way that it is m that remains unchanged i.e.

11.12

$$\mathcal{I} \sim s^{d/2} \left(1 + A(\Lambda, s) \frac{u_3^2}{m} + \dots \right)^{-1/2}$$

A more detailed study of such a case is done in 12.

But this redefinition forces all other parameters to zero and so again there is no continuous transition.

Finally m doesn't diverge for $d_c = 6$ even if $\mathcal{I}$ is chosen as indicated in §05 since in this case $A(\Lambda, s) \sim (1 - s^{-2})$ but we have already discussed this situation. The model that served as an example for the above considerations is the Potts-model (Po) discussed in the spirit of the R.G. by (AmSh), (Go), (Sz). General considerations on third order terms can be found in (Wa) and (ZiWä).

Then

$$05.23 \quad G_0^{*-1}(\vec{q}) = s^{2-\eta} [G_0^{*-1}(\vec{q}/s) + \Sigma_s^*(\vec{q}/s)]$$

We can then expand both sides in q^2

$$G_0^{*-1}(\vec{q}) = m^* + e_1^* q^2 + \dots$$

$$\Sigma_s^*(\vec{q}) = \Sigma_s^*(0) + q^2 \left(\frac{\partial \Sigma_s^*(q)}{\partial q^2} \right)_{q=0} + \dots$$

If we then fix $e_1^* = 1$ which can be done by a suitable choice of the units of q we find by comparison

$$05.24 \quad \eta = \log \left(1 + \left(\frac{\partial \Sigma_s^*(q)}{\partial q^2} \right)_{q=0} \right)$$

The condition $e_1^* = 1$ is one possible choice. There are situations where other choices are preferable (see § I1 and § I2).

In what follows the exponents will be calculated to lowest order in ϵ . This has the consequence to restrict very much the number of parameters involved in μ .

§ 06. Third-order couplings.

It is the object of the following chapters to use the techniques introduced in § 05 on models with third order terms.

In chapter I the coupling constants are taken to be wave vector independent in chapters II and III particular wave vector dependence of the third-order term is discussed.

The problem of third-order terms is well illustrated by the simple scalar model

$$06.1 \quad F\{\phi\} = \frac{1}{\lambda} \int d\vec{x} \left[m \phi^2(\vec{x}) + \sum_{i=1}^d \left(\frac{\partial}{\partial x_i} \phi(\vec{x}) \right)^2 \right] \\ + \int d\vec{x} u_3 \phi^3(\vec{x}) + \int d\vec{x} u_4 \phi^4(\vec{x})$$

The full free energy describes then the coupling of two different fields, a scalar one with a vector field.
Explicitely :

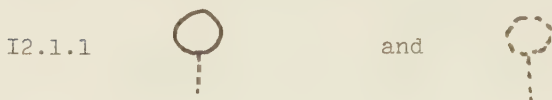
$$\begin{aligned}
 F = & \frac{1}{2} \int^{\wedge} (m_1 + c_1 q^2) \Phi_1(\vec{q}_1) \Phi_1(-\vec{q}_1) \\
 & + \frac{1}{2} \int^{\wedge} (m_2 + c_2 q^2) \sum_{n \neq 1}^{\wedge} \Phi_n(\vec{q}) \Phi_n(-\vec{q}) \\
 & + \int^{\wedge} w_3 \Phi_1(\vec{q}_1) \Phi_1(\vec{q}_2) \Phi_1(\vec{q}_3) \delta(\sum_{i=1}^3 \vec{q}_i) \\
 & + \int^{\wedge} u_3 \Phi_1(\vec{q}_1) \sum_{n \neq 1}^{\wedge} \Phi_n(\vec{q}_2) \Phi_n(\vec{q}_3) \delta(\sum_{i=1}^3 \vec{q}_i) \\
 & + \int^{\wedge} w_4 \Phi_1(\vec{q}_1) \Phi_1(\vec{q}_2) \Phi_1(\vec{q}_3) \Phi_1(\vec{q}_4) \delta(\sum_{i=1}^4 \vec{q}_i) \\
 & + \int^{\wedge} v_4 \Phi_1(\vec{q}_1) \Phi_1(\vec{q}_2) \sum_{n \neq 1}^{\wedge} \Phi_n(\vec{q}_3) \Phi_n(\vec{q}_4) \delta(\sum_{i=1}^4 \vec{q}_i) \\
 & + \int^{\wedge} u_4 \sum_{n \neq 1}^{\wedge} \Phi_n(\vec{q}_1) \Phi_n(\vec{q}_2) \Phi_n(\vec{q}_3) \Phi_n(\vec{q}_4) \delta(\sum_{i=1}^4 \vec{q}_i)
 \end{aligned}$$

The additional couplings are generated from I2.0.2, through renormalisation, by the principle that every type of coupling allowed by the symmetry of the original model is automatically generated.

§12.1 Dealing with the tadpoles.

In this technical paragraph we will set up a transformation which will eliminate the tadpoles in every step of the renormalisation procedure (NeFi).

If we associate a dotted line to the Φ_1 field and a continuous one to the $\Phi_{n \neq 1}$, it is immediate that two tadpole diagrams appear to lowest order :



They are equivalent to a external field in the Φ_1 direction i.e. to a term of the form $h \int \Phi_1(x) d\vec{x}$.

In studying the scaling behaviour of our model it is necessary to eliminate such terms. Let us then consider our initial free energy I2.0.3 as we have done on § 05 that is as a point in the parameter space :

I2.1.2
$$\mu = (m_1, e_1; m_2, e_2; w_3, u_3; w_4, u_4, u_4)$$

Then, after the first step of renormalisation

$$\mu \rightarrow \mu^1$$

I2.1.3
$$\mu^1 = (h; m_1^1, e_1^1; m_2^1, e_2^1; w_3^1, u_3^1; w_4^1, u_4^1, u_4^1)$$

where h corresponds to the tadpole term.
Since

$$\text{Solid Tadpole} = (N-1) u_3 \Phi_1(\vec{q}) \delta(\vec{q}) \int_{N_1} d\vec{p} \frac{1}{m_1 + e_1 p^2} \cong (N-1) \frac{u_3}{e_1} \Phi_1(\vec{q}) \delta(\vec{q}) \int_{N_1} d\vec{p} \frac{1}{p^2}$$

$$\text{Dashed Tadpole} = 3 w_3 \Phi_1(\vec{q}) \delta(\vec{q}) \int_{N_3} d\vec{p} \frac{1}{m_1 + e_1 p^2} \cong 3 \frac{w_3}{e_1} \Phi_1(\vec{q}) \delta(\vec{q}) \int_{N_3} d\vec{p} \frac{1}{p^2}$$

h is given in lowest order by

$$I2.1.4 \quad h = \left((N-1) \frac{u_2}{e_2} + 3 \frac{w_3}{e_2} \right) \int_{\Lambda_1}^1 d\vec{p} \frac{1}{p^2}$$

where s is again the scaling factor.

Of course μ' is a function of μ

$$I2.1.5 \quad \mu' = f_{\Lambda, s}(\mu)$$

At this stage h can be eliminated by a shift on ϕ_1

$$\phi_1 \rightarrow \gamma + \kappa$$

or

$$\phi_1(x) \rightarrow \gamma(x) + \kappa$$

and we get

$$I2.1.6 \quad \mu' = g_{\kappa}(\mu')$$

with

$$\mu' = (h + m_1' \kappa + 3 \kappa_3' (\kappa)^2 + 4 w_4' \kappa^3;$$

$$m_1' + 6 \kappa w_3' + 12 \kappa^2 w_4', \quad e_1';$$

$$m_2' + 2 \kappa u_3' + 4 \kappa^2 v_4', \quad e_2';$$

I2.1.7

$$w_3' + 4 \kappa w_4', \quad u_3' + 2 \kappa v_4';$$

$$w_4', \quad v_4', \quad u_4') =$$

$$= (h'; m_1', e_1'; m_2', e_2'; w_3', u_3'; w_4', v_4', u_4')$$

h is eliminated by fixing κ as the solution of

$$I2.1.8 \quad h + m_1' \kappa + 3 \kappa^2 w_3' + 4 \kappa^3 w_4' = 0$$

So the full recursion relations are a combination of I2.1.5 and I2.1.6

$$I2.1.9 \quad \mu' = g_k(f_{\Lambda, S}(\mu))$$

with μ given by I2.1.8

§ I2.2 Note on the anomalous dimension.

One would think that through the diagrams



there is a contribution to ϕ_1 and ϕ_2 of the order of u_1^2, u_3^2 . This would be interesting since for

$$u_1, u_3 \sim \epsilon^{1/2} + \mathcal{O}(\epsilon^{3/2})$$

these terms would introduce an anomalous dimension η of the order ϵ

But as it is explicitly shown in the Appendix A3 these diagrams don't depend on q , so the first diagram contributing to η is of the type  $\sim \epsilon^k$

On the other hand η is not automatically defined. We mentioned in I2.0 that we have to deal with two types of fields. This means that there is no a priori reason that ϕ_i scales the same way as ϕ_{m+1} . We are led to introduce two different field rescaling factors by

$$I2.2.1 \quad \begin{aligned} \phi_i &\rightarrow J_i \phi_i' \\ \phi_{m+1} &\rightarrow J_{m+1} \phi_{m+1}' \end{aligned}$$

But we know that η is ultimately computed through the J using relations like 05.23

So we should introduce two anomalous dimensions, η_1 and η_2

§ I2.3 Computations.

Before establishing explicitly the recursion relations of our coupling constants let us make some qualitative statements related to m_1 and e_1 .

We assume that we can put $e_2 = 1$. Then the recursion relations of m_1 and e_1 are of the form:

$$\begin{aligned} I2.3.1 \quad m_1' &= J_1^2 s^{-d} (m_1 - 2 u_3^2 (N-1)) \int_{N/2}^{\infty} \frac{d\vec{p}}{m_1 + p^2} + \\ &\quad + \mathcal{O}(v_4, u_4, w_3^2) + \dots \\ e_1' &= J_1^2 s^{-d-2} (e_2 + \dots) \end{aligned}$$

Generally the value of J_1 is determined by fixing e_1 to a given value (for example one, see §05).

This yields $J_1 = s^{1+d/2}$

For this choice it becomes clear from I2.3.1 that as the recursion is repeated, or $s \rightarrow \infty$, $m_1 \rightarrow \infty$, too. Explicitly, since

$$\begin{aligned} I2.3.2 \quad \int_{N/2}^{\infty} \frac{d\vec{p}}{(2\pi)^d} p^{-4} &= \frac{1}{2^{3/2}} \log s \\ m_1' &= -2 u_3^2 \frac{N-1}{2^{3/2}} \frac{s^2 \log s}{1-s^2} + \mathcal{O}(u_4 \dots) \end{aligned}$$

One way out is to put $m_1^* = 0$ which leads to $u_3^* = 0$ and eventually to $w_4^* = v_4^* = w_3^* = 0$

What remains is a Heisenberg model with $N-1$ components with an obvious lack of interest as far as our model is concerned. But there is another possibility to neutralize m_1 and this is to fix J_1 in such a way that m_1 remains unchanged. In this case the Φ_1 part of the model works like a background in which the Φ_{m+1} field becomes critical.

Explicitely this is done by choosing

$$J_1 = s^{d/2} \left(1 - 2(N-1) \frac{u_2^2}{m_1} \int_{N_1}^1 d\vec{p} p^{-4} + \dots \right)^{-1/2}$$

I2.3.3

$$\cong s^{d/2} \left(1 + (N-1) \frac{u_2^2}{m_1} \int_{N_1}^1 d\vec{p} p^{-4} + \dots \right)$$

Injecting this choice into I2.3.1 we see that

$$m_1' = m_1 \quad \text{and} \quad e_1' = s^{-2} (e_1 + \dots) \rightarrow 0$$

This choice of J_1 has important influence on the other coupling parameters.

Since

$$\begin{aligned} \text{a.} \quad w_3' &\sim J_1^3 s^{-2d} (w_3 + \dots) \\ &= s^{-2+\epsilon/2} (w_3 + \dots) \end{aligned}$$

it follows that $w_3 \rightarrow 0$

$$\begin{aligned} \text{b.} \quad v_4' &\sim J_1^2 J_2^2 s^{-3d} (v_4 + \dots) \\ &= s^{-2+\epsilon} (v_4 + \dots) \end{aligned} \quad \text{for } J_2 = s^{1+\frac{d}{2}}$$

it follows that $v_4 \rightarrow 0$

$$\begin{aligned} \text{c.} \quad w_4' &\sim J_1^4 s^{-3d} (w_4 + \dots) \\ &= s^{-4+\epsilon} (w_4 + \dots) \end{aligned}$$

it follows that $w_4 \rightarrow 0$

So not only w_3, v_4, w_4 cannot upset the stability of the possible fixed point but they tend to zero, too. They are irrelevant parameters and we can drop them together with e_1 from our free energy I2.0.3

What happens then to the tadpole-produced field h discussed in I2.1 ?

We see that by I2.1.7

$$\text{I2.3.4} \quad h' = h + m_1 \kappa \quad \text{with} \quad h = \frac{u_2}{e_2} (N-1) \int_{\Lambda/S} \frac{d\vec{p}}{(2\pi)^d} p^{-2}$$

and I2.1.8 becomes

$$\text{I2.3.5} \quad h + m_1 \kappa = 0 \rightarrow \kappa = -\frac{h}{m_1} = -(N-1) \frac{u_2}{m_1 e_2} \int_{\Lambda/S} \frac{d\vec{p}}{(2\pi)^d} p^{-2}$$

Incidentally we can again choose $e_2 = 1$

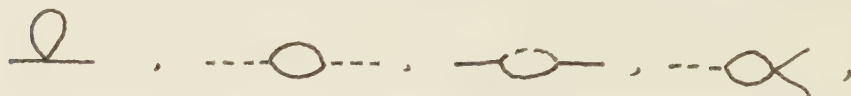
Then μ' is given by

$$\begin{aligned} \mu' &= (m_1, 0; m_2' - 24 \frac{u_3'}{m_1}, 1; 0, u_3'; 0, 0, u_4') \\ &= (m_1', 0; m_2', 1; 0, u_3'; 0, 0, u_4') \end{aligned}$$

and the effective free energy by

$$\begin{aligned} \text{I2.3.6} \quad F &= \frac{1}{2} \int^{\Lambda} m_1 \phi_1(\vec{q}) \phi_1(-\vec{q}) \\ &+ \frac{1}{2} \int^{\Lambda} (m_2 + q^2) \sum_{n \neq 1}^N \phi_n(\vec{q}) \phi_n(-\vec{q}) \\ &+ \int^{\Lambda} u_3 \phi_1(\vec{q}_1) \sum_{m \neq 1}^N \phi_m(\vec{q}_2) \phi_m(\vec{q}_3) \delta(\sum_{i=1}^3 \vec{q}_i) \\ &+ \int^{\Lambda} u_4 \sum_{n, m \neq 1}^N \phi_n(\vec{q}_1) \phi_n(\vec{q}_2) \phi_m(\vec{q}_3) \phi_m(\vec{q}_4) \delta(\sum_{i=1}^4 \vec{q}_i) \end{aligned}$$

The diagrams that we need to calculate are the following



Examples of diagram calculations are given in the appendices A1-3.

Proceeding now as indicated in §I2 we can write the following equations

$$m_2' = J_2^2 s^{-d} (m_2 + 2 \text{tadpole} - \text{self-energy loop} + \dots)$$

$$u_3' = J_1 J_2^2 s^{-2d} (u_3 - \text{vertex correction} + \frac{1}{3!} \text{triangle} - \dots)$$

$$u_4' = J_2^4 s^{-3d} (u_4 - \frac{1}{2} \text{bubble} + \frac{1}{2} \text{crossed bubble} -$$

and

$$- \frac{1}{4!} \text{box} + \dots)$$

$$m_2' = m_2^2 + 2 u_3' \kappa$$

$$u_3' = u_3'$$

$$u_4' = u_4'$$

As we have seen before

$$J_1 = s^{d/2} (1 + \frac{1}{2m_1} \text{self-energy loop} - \dots)$$

and

$$J_2 = s^{1+d/2}$$

chosen to keep e_2 fixed.

With this choice and the actual calculation of the diagrams the transformation

$$\mu' = g_k(f_{n,s}(\mu))$$

is explicitly given

by

$$m_2' = s^2 \left(m_2 + 4(N+1)g_{01}u_4 - 2(N+1)\frac{u_3^2}{m_1}g_{01} - \right. \\ \left. - 4(N+1)m_2u_4K_4\log s + 2(N+1)\frac{u_3^2}{m_1}m_2K_4\log s \right)$$

$$u_3' = s^{6/2} \left(u_3 - 4(N+1)u_3u_4K_4\log s + \right. \\ \left. + (N+3)\frac{u_3^3}{m_1}K_4\log s \right)$$

$$u_4' = s^6 \left(u_4 - 4(N+7)u_4^2K_4\log s + \right. \\ \left. + 24u_4\frac{u_3^2}{m_1}K_4\log s - 4\frac{u_3^4}{m_1^2}K_4\log s \right)$$

with

$$g_{01} = \int_{\Lambda_s}^{\Lambda} \frac{d\vec{p}}{(2\pi)^d} p^{-2} = \frac{K_4}{2} \Lambda^2(1-s^{-2}) + \mathcal{O}(\epsilon)$$

$$K_4 = (2^3\pi^2)^{-1}$$

The linearised equation is

$$\delta \mu' = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ 0 & M_{22} & M_{23} \\ 0 & M_{32} & M_{33} \end{pmatrix} \delta \mu$$

with

$$M_{11} = S^{2[1+(N+1)(x^2-2y)]} u_4; \quad M_{12} = S^2 h_1(N); \quad M_{13} = S^2 h_3(N);$$

$$M_{22} = S^{\epsilon/2 + (3(N+3)x^2 - 4(N+1)y)} u_4; \quad M_{23} = -S^{\epsilon/2} 4(N+1) u_3^* u_4 \log S$$

$$M_{33} = S^{\epsilon + (24x^2 - 8(N+7)y)} u_4; \quad M_{32} = \frac{16}{u_3^*} S^{\epsilon} (3x^2 y - x^4) u_4 \log S$$

with $x^2 = \frac{u_3^{*2}}{m}$, $y = u_4^*$

$h_1(N)$ and $h_2(N)$ being functions of N which play no role in the diagonalisation.

We know already that

$$\eta_2 \equiv \eta = 0 + \mathcal{O}(\epsilon^2)$$

From the diagonalisation of the matrix it follows that

$$\nu = \frac{1}{2} - (N+1)(x^2 - 2y) u_4 + \mathcal{O}(\epsilon^2)$$

Turning back to the equations of I15 we may now list the solutions.

F1) $u_3^* = 0; \quad u_4^* = 0 \quad \rightarrow \quad \nu = \frac{1}{2}$

is a "Gaussian" solution which is of course unstable.

F2) $u_3^* = 0; \quad u_4^* = \frac{\epsilon}{4(N+7)} \cdot u_4^{-1}$

is the Heisenberg solution.

Remember that $N > 1$ by construction. For $N = 1$ the third order term drops together with all the $\{\Phi_{m \neq 1}\}$ and we have an Ising model.

For this solution

$$\begin{aligned} M_{11} &= S^{2 + \frac{N+1}{N+7} \epsilon} \\ M_{22} &= S^{\epsilon/2 \frac{5-N}{N+7}} ; M_{23} = 0 \\ M_{33} &= S^{-\epsilon} ; M_{32} = 0 \end{aligned}$$

The corresponding eigenvalues are

$$\lambda_1 = S^{\epsilon/2 \frac{5-N}{N+7}} ; \lambda_2 = S^{-\epsilon}$$

One would expect this solution to be unstable for every N . Apparently this is not the case since for $N > 5$ $\lambda_1 < 1$. The interpretation could be that for N big enough the inhibiting component Φ_2 becomes submerged by the $\{\Phi_{m \neq 1}\}$ fluctuations.

$$F3) \quad x^2 = \frac{5-N}{2(N-1)(N+7) \kappa_4} \epsilon ; \quad y = \frac{1}{(N-1)(N+7) \kappa_4} \epsilon$$

This solution is acceptable for $N \leq 5$ because otherwise becomes imaginary and the coupling corresponds to a dissipative term. It is interesting that $N = 5$ is again a borderline.

The critical exponent ν is

$$\nu = \frac{1}{2} + \frac{N+1}{4(N+7)} \epsilon$$

which is a Heisenberg exponent. To find out if this solution is a "real Heisenberg" we should check its stability we find

$$\lambda_1 = S^{-\epsilon \frac{5-N}{N+7}} \quad \lambda_2 = S^{-\epsilon}$$

This means that in the region $N \leq 5$ this solution is stable i.e. a normal Heisenberg solution.

F4)

$$x^2 = \frac{1}{2 K_4 (N-1)} \quad ; \quad y = \frac{1}{4 K_4 (N-1)}$$

The exponent ν is

$$\nu = \frac{1}{2}$$

and the eigenvalues are

$$\lambda_1 = s^{\epsilon} \quad ; \quad \lambda_2 = s^{-\epsilon}$$

To give an interpretation of this Gaussian fixed point which should correspond to a tricritical point, let us consider again I2.3.6

$$\begin{aligned} F = & \frac{1}{2} \int^1 m_1 \phi_1(\vec{q}_1) \phi_1(\vec{q}_2) \delta(\vec{q}_1 + \vec{q}_2) \\ & + \frac{1}{2} \int^1 (m_2 + q^2) \sum_{n \neq 1}^N \phi_n(\vec{q}) \phi_n(-\vec{q}) \\ & + \int^1 u_3 \phi_1(\vec{q}_1) \sum_{n \neq 1}^N \phi_n(\vec{q}_2) \phi_n(\vec{q}_3) \delta(\sum_{i=1}^3 \vec{q}_i) \\ & + \int^1 u_4 \sum_{n, m \neq 1}^N \phi_n(\vec{q}_1) \phi_n(\vec{q}_2) \phi_m(\vec{q}_3) \phi_m(\vec{q}_4) \delta(\sum_{i=1}^4 \vec{q}_i) \end{aligned}$$

An appropriate shift in the ϕ_i field, that is

$$\phi_i(\vec{x}) \rightarrow \phi_i(\vec{x}) + \frac{u_3}{m_1} \sum_{n \neq i}^N \phi_n^2(\vec{x}) = \psi(\vec{x})$$

decouples the free energy in

$$\begin{aligned} F = & \frac{1}{2} m_1 \int^1 \psi(\vec{q}) \psi(-\vec{q}) \\ & + \frac{1}{2} \int^1 (m_1 + q^2) \sum_{n \neq i}^N \phi_n(\vec{q}) \phi_n(-\vec{q}) \\ & + \left(u_4 - \frac{u_3^2}{2m_1} \right) \int \sum_{n \neq m \neq i}^N \phi_n(\vec{q}_1) \phi_m(\vec{q}_2) \phi_m(\vec{q}_3) \phi_n(\vec{q}_4) \delta\left(\sum_{i=1}^4 \vec{q}_i\right) \end{aligned}$$

But we know that a continuous phase transition can take place as long as

$$u_4 - \frac{u_3^2}{2m_1} \geq 0$$

the equality holding for the tricritical case.

Recall now the expression for

$$\nu = \frac{1}{2} - \frac{N+1}{2} (x^2 - 2\gamma) u_4$$

But $x^2 = \frac{u_3^{*2}}{m_1}$ and $\gamma = u_4^*$

so $\nu = \frac{1}{2} - (N+1) \left(\frac{u_3^{*2}}{2m_1} - u_4^* \right) u_4$

In our case F4

$$x^2 = 2\gamma \quad \text{that is} \quad u_4^* - \frac{u_3^{*2}}{2m_1} = 0$$

So F4 corresponds really to a tricritical point.

CHAPTER II

ON MODELS WITH WAVE-VECTOR DEPENDENT
THIRD ORDER COUPLING PARAMETER§ III Linear wave-vector dependence§ III.0 Dealing with the tadpoles.

We have seen in § II.2 that, in the case of constant third order coupling parameter, we need a special transformation to eliminate the tadpole diagrams. In this paragraph the third order term is assumed linear in the wave-vector, that is of the form

$$\text{III.0.1} \quad \int \sum_{\alpha\beta\gamma} (\vec{n}, \vec{q}) u_{3\alpha\beta\gamma} \phi_{\alpha}(\vec{q}_1) \phi_{\beta}(\vec{q}_2) \phi_{\gamma}(\vec{q}_3) \delta(\sum_{i=1}^3 \vec{q}_i)$$

where $\vec{n}$ is a unit vector in q-space and, for the time being, $u_{3\alpha\beta\gamma}$ are imaginary. A term like this corresponds to a coupling between the field components and their first order derivatives. Note that although the $u_{3\alpha\beta\gamma}$ are arbitrary not every term like III.0.1 is interesting. For example a term like

$$\text{III.0.2} \quad \int \left(\frac{\partial}{\partial x_i} \hat{\phi}(\vec{x}) \right) \hat{\phi}^2(\vec{x}) = -i \int q_i \phi(\vec{q}_1) \phi(\vec{q}_2) \phi(\vec{q}_3) \delta(\sum_{i=1}^3 \vec{q}_i)$$

is zero since it is just a surface integral. We assume now that the free energy is given by II.0.1 plus an even polynomial in the field components with coupling parameters that do not have any odd q-dependence. It is easily seen that the lowest order tadpole is zero. In fact it is either of the form

$$\text{III.0.3} \quad \text{---} \circ \sim (\vec{n}, \vec{q}) \delta(\vec{q}) \int \frac{d\vec{p}}{(2\pi)^d} G_0(\vec{p})$$

or

$$\text{---} \circ \sim \delta(\vec{q}) \int \frac{d\vec{p}}{(2\pi)^d} (\vec{n}, \vec{p}) G_0(\vec{p})$$

as long as $G_0(\vec{p}) = G_0(-\vec{p})$

We have associated a dotted line to the field component which is multiplied by $(\vec{n}, \vec{q})$ and this will be the case throughout this chapter. But we need some argument to show that tadpoles are zero to every order. This is so because we know that external fields change the critical behaviour (eventually they totally inhibit it) and hence would be relevant independently of their order.

To show this we note that our free energy is invariant under complex conjugation because

a. $\phi_q^*(\vec{q}) = \phi_q(-\vec{q})$ because $\phi(\vec{x})$ is chosen real

b. $u_{3\alpha\beta\gamma}$ is imaginary.

On the other hand consider the tadpoles. They are either of the form

III.0.4 1)



or 2)



The first type is trivially zero as in III.0.3. The second yields a term which is odd in the u_3 's that is of the form

$$w \phi_q(0) = w \int \phi_q(\vec{x}) d\vec{x}$$

But $w^* = -w$, so such a term changes sign under complex conjugation which is a symmetry of the original model. But the R.G. does not break symmetries (WiKo). So the tadpoles must vanish. Actually since we want to state this matter of fact to any order we should mention first that this type of argument applies to every vertex in the following sense :

a. all odd vertices produced by the renormalisation can have only an odd q -dependence.

b. all even vertices can have only an even q -dependence.

Relaxing the condition that the u_3 's should be imaginary means that we introduce in the model terms which describe dissipation. A priori there is no reason why such terms should not produce tadpoles. One has to consider the form of the graphs which could produce tadpoles to show that they are zero. This is certainly the case for the model we discuss later.

III.1.1 A model with anomalous, anomalous dimension.

We wish now to set up a model with a third order parameter which is linear in the wave vector. On the other hand this dependence should not be trivial in the sense of III.0.2. So we choose a third order term of the form

$$\text{III.1.1} \quad \int^{\Lambda} \sum_{n \neq 1} u_3 q_i^k \phi_i(\vec{q}_1) \phi_n(\vec{q}_2) \phi_n(\vec{q}_3) \delta(\sum_{i=1}^3 \vec{q}_i)$$

where q_i^k means the k -th component of $\vec{q}_i$. We add this term to a Heisenberg free energy:

$$\text{III.1.2} \quad F_0 = \frac{1}{2} \int^{\Lambda} (m + c q^2) \sum_{n=1}^N \phi_n(\vec{q}) \phi_n(-\vec{q}) \\ + \int^{\Lambda} u_4 \sum_{n,m=1}^N \phi_n(\vec{q}_1) \phi_n(\vec{q}_2) \phi_n(\vec{q}_3) \phi_m(\vec{q}_4) \delta(\sum_{i=1}^4 \vec{q}_i)$$

But III.1.1 introduces an anisotropy in the system and this has of course an influence on the quadratic part. On the other hand it introduces a distinction between ϕ_i and $\{\phi_{n \neq 1}\}$. This leads to

$$\langle \phi_i^2 \rangle \neq \langle \phi_{n \neq 1}^2 \rangle$$

and hence to mass splitting. We note that $n = 1$ is not included in III.1.2 again because III.0.2

It follows from the above comments that the free energy should be

$$\text{III.1.3} \quad F_1 = \frac{1}{2} \int^{\Lambda} G_{01}^{-1}(\vec{q}) \phi_1(\vec{q}) \phi_1(-\vec{q}) \\ + \frac{1}{2} \int^{\Lambda} G_{02}^{-1}(\vec{q}) \sum_{n \neq 1}^N \phi_n(\vec{q}) \phi_n(-\vec{q}) \\ + \int^{\Lambda} u_3 q_i^k \phi_i(\vec{q}_1) \sum_{n \neq 1}^N \phi_n(\vec{q}_2) \phi_n(\vec{q}_3) \delta(\sum_{i=1}^3 \vec{q}_i) \\ + \int^{\Lambda} u_4 \sum_{n,m=1}^N \phi_n(\vec{q}_1) \phi_n(\vec{q}_2) \phi_n(\vec{q}_3) \phi_m(\vec{q}_4) \delta(\sum_{i=1}^4 \vec{q}_i)$$

with

$$G_{01}^{-1}(\vec{q}) = m_1 + e_1 q^2 + t_1 (q^4)^2$$

III.1.4

$$G_{02}^{-1}(\vec{q}) = m_2 + e_2 q^2 + t_2 (q^4)^2$$

In diagrammatic language these corrections stem

from



and



with the dotted lines for ϕ_1 and the continuous ones for $\{\phi_{n \neq 1}\}$

But there are other terms that can be added without altering the symmetry of this model. They are the fourth order terms,

$$\begin{aligned} & \int w_4 \phi_1(\vec{q}_1) \phi_1(\vec{q}_2) \phi_1(\vec{q}_3) \phi_1(\vec{q}_4) \delta(\sum_{i=1}^4 \vec{q}_i) \\ & \int v_4 \phi_1(\vec{q}_1) \phi_1(\vec{q}_2) \sum_{n \neq 1}^N \phi_n(\vec{q}_3) \phi_n(\vec{q}_4) \delta(\sum_{i=1}^4 \vec{q}_i) \end{aligned}$$

III.1.6

So the full free energy is given by

$$\begin{aligned} F = & \frac{1}{2} \int G_{01}^{-1}(\vec{q}) \phi_1(\vec{q}) \phi_1(-\vec{q}) \\ & + \frac{1}{2} \int G_{02}^{-1}(\vec{q}) \sum_{n \neq 1}^N \phi_n(\vec{q}) \phi_n(-\vec{q}) \\ & + \int \sum_{n \neq 1}^N u_3 q^4 \phi_1(\vec{q}_1) \phi_n(\vec{q}_2) \phi_n(\vec{q}_3) \delta(\sum_{i=1}^3 \vec{q}_i) \\ & + \int w_4 \phi_1(\vec{q}_1) \phi_1(\vec{q}_2) \phi_1(\vec{q}_3) \phi_1(\vec{q}_4) \delta(\sum_{i=1}^4 \vec{q}_i) \\ & + \int v_4 \phi_1(\vec{q}_1) \phi_1(\vec{q}_2) \sum_{n \neq 1}^N \phi_n(\vec{q}_3) \phi_n(\vec{q}_4) \delta(\sum_{i=1}^4 \vec{q}_i) \\ & + \int u_4 \sum_{n \neq 1}^N \phi_n(\vec{q}_1) \phi_n(\vec{q}_2) \phi_n(\vec{q}_3) \phi_n(\vec{q}_4) \delta(\sum_{i=1}^4 \vec{q}_i) \end{aligned}$$

III.1.7

We will return to III.1.6 at a later stage. Since $G_{0i}(\vec{q}) = G_{0i}(-\vec{q})$ we know, § III.0, that no tadpoles can appear. In fact this would be the case even if u_3 would be complex, because a diagram like



that is a term proportional to $\phi_{n\pm 1}(0)$ would break the symmetry

$$\phi_{n\pm 1}(0) = \int \hat{\phi}_{n\pm 1}(\vec{x}) d\vec{x}$$

$$\phi_{n\pm 1} \rightarrow -\phi_{n\pm 1}$$

For this reason we will keep in mind through this paragraph that u_3 can be in general complex although our main interest is the case u_3 imaginary.

We can now make some qualitative statements on our model. Consider first the situation $u_4 = u_5 = 0$ in III.1.7 (we will see that these are possible fixed points).

We distinguish then two cases :

a) $m_1 \neq 0$

We then find

$$\begin{aligned} F = & \frac{1}{2} \int^1 (m_1 + e_1 q^2 + t_1 (q^4)^2) \chi(\vec{q}) \chi(-\vec{q}) \\ & + \frac{1}{2} \int^1 (m_2 + e_2 q^2 + t_2 (q^4)^2) \sum_{n \neq 1}^N \phi_n(\vec{q}) \phi_n(-\vec{q}) \\ & + \int^1 u_4 \sum_{n, n \neq 1}^N \phi_n(\vec{q}_1) \phi_n(\vec{q}_2) \phi_n(\vec{q}_3) \phi_n(\vec{q}_4) \delta(\sum_{i=1}^4 \vec{q}_i) \end{aligned}$$

+ terms of the type $(q^4)^3 \chi \phi^2 q^2 \phi^2$ which we neglect since they are irrelevant.

To find II.1.8 we just "completed the squares" after a partial integration in the third order term. We need the condition $m_1 \neq 0$ because the terms listed as irrelevant are proportional to powers of $\frac{u_2}{m_1}$.

The first part of F in II.1.8 is just an "anisotropic Gaussian" model (Ah) which can be integrated out leaving a Heisenberg model in the $\{\phi_{n \neq 1}, \vec{q}\}$. We see that in this case the third order term has no influence.

b) $m_1 = 0$

In this case we can look at F either as a Lagrangian describing a coupling between a "mass-less scalar field" and a "massive vector field" or we can say that the field ϕ_i has undergone a Gaussian transition, determined by $m_1 = 0$.

Unfortunately, but not surprisingly, we cannot eliminate the third order term this time. In fact the shift that would permit this elimination should be :

$$\frac{\partial}{\partial x^i} \phi_i(\vec{x}) \rightarrow \frac{\partial}{\partial x^i} \tilde{\chi}(\vec{x}) + \frac{u_3}{t_1} \sum_{n \neq 1}^N \tilde{\phi}_n^2(\vec{x})$$

or

$$g^k \phi_k(\vec{q}) \rightarrow g^k \chi(\vec{q}) + i \frac{u_3}{t_1} \sum_{n \neq 1}^N \phi_n(\vec{q}_1) \phi_n(\vec{q}_2) \delta(\vec{q}_1 + \vec{q}_2)$$

But through the term

$$\sum_i \left(\frac{\partial}{\partial x^i} \tilde{\phi}_i(\vec{x}) \right)^2$$

the third order term is reproduced.

Finally there is again not much point in trying to eliminate the third order term in the case where at least $w_4 \neq 0$ since any elimination attempt introduces new terms like other third order terms, fifth order terms etc. which are more complicated to handle as the original third order term.

A further problem is introduced by the parameters t_1 and t_2 . A clue in handling them is given in (NeFi). In fact if we consider the case $N = 1$, III.1.7 reduces to an anisotropic Ising model and t_1 is then a marginal variable which controls the critical scattering but has no influence on the critical exponents. This remains so for $N \neq 1$. So we will not try to calculate any fixed points t_1^* and t_2^* . More explicit statements about t_1 and t_2 can be found in the appendix A6. What we will do is to keep t_1 and t_2 in the free propagator without handling them as expansion parameters. Again they will drop from the critical exponents.

For the establishment of the recursion relations there are many graphs that are needed. They are listed with their values in appendix A4 and an explicit calculation of the most important ones

i.e.  and  is given in appendix A5

Further we don't consider one-particle reducible graphs. This is again explained in the appendix A4.

§ III.2 The recursion relations.

Following the path introduced in § 0.5 the previous paragraph and the appendices A4-A6 we can establish the following equations

$$a. \quad e_1' = T_1^2 s^{-d-2} e_1 + \dots$$

$$m_1' = T_1^2 s^{-d} \left\{ m_1 + 12 u_4 g_{0,0}^{1,0} + 2 u_4 g_{0,0}^{0,0} \delta_{N,1} \right.$$

III.2.1 b.

$$\left. - 12 u_4 m_1 g_{0,0}^{2,0} - 2 m_2 u_4 g_{0,0}^{0,2} \delta_{N,1} + \dots \right\}$$

$$u_3' = T_1 T_2^2 s^{-2d-1} \left\{ u_3 - 4(N+1) u_3 u_4 g_{0,0}^{0,0} \delta_{N,1} \right.$$

c.

$$- 2 u_3 u_4 (g_{0,0}^{1,1} + g_{2,2,0}^{1,2} (e_2 + t_2)) \delta_{N,1}$$

$$- 8 u_3^2 g_{2,2,0}^{1,2} \delta_{N,1} + \dots \left. \right\}$$

a.
$$e_2' = J_2^2 s^{-d-2} \left\{ e_2 + \right.$$

$$+ 4u_3^2 \left[-\frac{1}{2} (e_1 g_{2k,0}^{21} + e_3 g_{2k,0}^{12}) + \right.$$

$$\left. + 2 (e_1^2 g_{2k,2i}^{31} + e_3^2 g_{2k,2i}^{13}) \right] \delta_{N,1} + \dots \left. \right\}$$

III.2.2

b.

$$m_2' = J_2^2 s^{-d} \left\{ m_2 + 4(N+1) g_{0,0}^{01} u_4 \delta_{N,1} + \right.$$

$$+ 2 g_{0,0}^{10} v_4 \delta_{N,1} - 4(N+1) g_{0,0}^{02} m_3 u_4 \delta_{N,1}$$

$$- 2 g_{0,0}^{12} m_1 v_4 \delta_{N,1} + 4u_3^2 g_{2k,0}^{11} \delta_{N,1}$$

$$\left. - 4 g_{2k,0}^{21} m_1 u_3^2 \delta_{N,1} - 4 g_{2k,0}^{12} m_2 u_3^2 \delta_{N,1} + \dots \right\}$$

c.

$$u_4' = J_2^4 s^{-3d} \left\{ u_4 - 4(N+7) u_4^2 g_{0,0}^{02} \delta_{N,1} \right.$$

$$- v_4^2 g_{0,0}^{20} \delta_{N,1} - 24 u_3^2 u_4 g_{2k,0}^{12} \delta_{N,1}$$

$$- 4 u_3^2 v_4 g_{2k,0}^{21} \delta_{N,1} -$$

$$\left. - 4 u_3^4 g_{4k,0}^{22} \delta_{N,1} \right\}$$

$$a. \quad w_4' = J_1^4 s^{-3d} \left\{ w_4 - 36 w_4^2 g_{0,0}^{0,2} - v_4^2 g_{0,0}^{0,2} \delta_{N,1} + \dots \right\}$$

III.1.1.3

$$b. \quad v_4' = J_1^2 J_2^2 s^{-3d} \left\{ v_4 - 8 v_4^2 g_{0,0}^{1,1} \delta_{N,1} - \right. \\ \left. - 12 v_4 w_4 g_{0,0}^{2,0} \delta_{N,1} - 4(N+1) u_4 v_4 g_{0,0}^{0,2} \delta_{N,1} + \dots \right\}$$

with the following meaning of the symbols :

$$\delta_{N,1} = 1 - \delta_{N,1}$$

$$g_{n_i, m_j}^{s, t} = \int_{1, s}^1 \frac{d\vec{p}}{(2\pi)^d} (p_i)^n (p_j)^m (G_{0,1}(\vec{p}))^s (G_{0,2}(\vec{p}))^t \Big|_{m_i=0}$$

We note that we didn't expand in e or in t . They appear in the equations because we expanded in the external wave-vector,

The equations have been calculated to lowest order in $\epsilon = 4 - d$ following the assumptions of II i.e.

$$m_1, m_2, w_4, v_4, u_4 \sim \epsilon$$

$$u_3 \sim \epsilon^{1/2}$$

We will now discuss these equations for different cases.

§ III.1.3 The case $w_4 = v_4 = 0$

This case was mentioned in III.1.1.3 where two possibilities were introduced (case a. and b. on page II5).

Let us rewrite the recursion equations for this case

$$a. \quad e_1' = J_1^2 s^{-d-2} e_1,$$

$$b. \quad m_1' = J_1^2 s^{-d} m_1,$$

III.1.3.1

$$u_3' = J_1 J_2^2 s^{-2d-1} \left\{ u_3 - 4(N+1) u_3 u_4 g_{0,0}^{0,2} \delta_{N,1} \right. \\ \left. - 8 u_3^2 g_{0,0}^{1,2} \delta_{N,1} + \dots \right\}$$

c.

$$a. \quad e_2' = J_2^2 s^{-d-2} \left\{ e_2 + \right.$$

$$+ 4u_3^2 \left[-\frac{1}{2} (e_1 g_{2k_0}^{21} + e_2 g_{2k_0}^{12}) + \right. \\ \left. + 2 (e_1^2 g_{2k_0}^{21} + e_2^2 g_{2k_0}^{12}) \right] \delta_{N,1} + \dots \left. \right\}$$

$$b. \quad m_2' = J_2^2 s^{-d} \left\{ m_2 + 4(N+1) g_{00}^{01} u_4 \delta_{N,1} \right.$$

$$+ u_3^2 g_{2k_0}^{11} \delta_{N,1} - 4(N+1) g_{00}^{02} m_2 u_4 \delta_{N,1} \\ \left. - 4 g_{2k_0}^{21} m_1 u_3^2 \delta_{N,1} - 4 g_{2k_0}^{12} m_2 u_3^2 \delta_{N,1} + \dots \right\}$$

III.3.2

$$c. \quad u_4' = J_2^4 s^{-3d} \left\{ u_4 - 4(N+7) u_4^2 g_{00}^{02} \delta_{N,1} \right.$$

$$\left. - 24 u_3^2 u_4 g_{2k_0}^{12} \delta_{N,1} - 4 u_3^4 g_{2k_0}^{22} + \dots \right\}$$

Looking at III.3.1 a. and b. we notice that we have two possibilities. Either we choose $J_1 = s^{1+1/2}$ such that e_1 remains a constant ($e_1 = 1$). Then

$$m_1' = s^2 m_1 \rightarrow \infty \quad \text{as} \quad s \rightarrow \infty$$

and we have to choose $m_1 = 0$

Or in order to avoid this we choose $J_1 = s^{d/2}$ such that m_1 remains a constant and

$$e_1' = s^{-2} e_1 \rightarrow 0 \quad \text{as} \quad s \rightarrow \infty$$

Let us consider first the second case which corresponds to the case b. of page II5.

From III.3.2 a. we may fix

$$J_2 = s^{1+d/2} (1 - C u_3^2)^{1/2} \approx s^{1+d/2} (1 + \frac{C}{2} u_3^2 + \dots)$$

such that e_2 remains unchanged. Actually $C \sim \log s$ (see appendix A5)
Then III.3.1 c. has the form

$$\begin{aligned} u_3' &= s^{-d/2+1} (1 + \frac{C}{2} u_3^2) (u_3 + a u_3 u_4 + b u_3^3 + \dots) \\ &= s^{-1+e/2} (u_3 + A u_3 u_4 + B u_3^3 + \dots) \end{aligned}$$

and it follows that the only possible solution is $u_3^* = 0$
since a solution $u_3^* \neq 0$ should satisfy

$$D u_3^{*2} + B u_4^* = s + O(\epsilon)$$

which is impossible in a power series sense (see § II).

But if $u_3^* = 0$ we are left with $J_2 = s^{1+d/2}$,

$$\begin{aligned} m_2' &= s^2 (m_2 + 4(N+1) g_{00}^{01} u_4 \delta'_{N,1} \\ &\quad - 4(N+1) g_{00}^{02} m_2 u_4 \delta_{N,1} + \dots) \end{aligned}$$

$$u_4' = s^2 (u_4 - 4(N+7) u_4^2 g_{00}^{02} \delta'_{N,1} + \dots)$$

which is the usual Heisenberg model situation arriving at the same conclusions as in page II5. The first case $m_1 = 0$ is more complicated. We assume that Φ_1 has undergone a "Gaussian" transition with $\eta_1 = 0$, $\chi_1 = 1/2$.

We will choose the J 's such that $e_1 = e_2 = 1$ and further we will set $t_1 = t_2 = t$. To justify this last step we remind that t_1 is a marginal variable which we cannot calculate from a recursion relation. So we choose its value to be equal t_2

These two conditions imply great simplifications because now

$$g^{st} = g^{ts}$$

since $G_{01}(\vec{p}) / m_{1=0} = G_{02}(\vec{p}) / m_{2=0} = \frac{1}{p^2 + t(p^4)^2}$

We then define

$$g_{24,2\ell}^{s,t} = A_{24,2\ell}^{s,t} \log s$$

where A is given by

$$g_{24,0}^{12} = \frac{1}{4} \mathcal{H}_4 (1+t)^{-3/2} \log s = g_{24,0}^{2,0}$$

$$g_{24,2\ell}^{13} = \frac{1}{24} \mathcal{H}_4 (1+t)^{-3/2} = g_{24,2\ell}^{3,1} \quad \ell \neq i$$

etc. see appendix A4

From the equation III.3.2 a. we find using the above relations

$$\begin{aligned} \text{III.3.3} \quad \mathcal{I}_2 &= s^{1+d/2} \left(1 + \frac{2}{3} u_3^2 A_{24,0}^{12} \log s + \dots \right) \\ &\approx s^{1+d/2} + \frac{2}{3} u_3^2 A_{24,0}^{12} + \dots \end{aligned}$$

which gives

$$\text{III.3.4} \quad s^{2-\eta_2} = \mathcal{I}_2^2 s^{-d} = s^{2+\frac{4}{3} u_3^{*2} A_{24,0}^{12}}$$

$$\text{III.3.5} \quad \eta_2 = -\frac{4}{3} u_3^{*2} A_{24,0}^{12} = -\frac{1}{3} \mathcal{H}_4 (1+t)^{-\frac{3}{2}} u_3^{*2}$$

It follows that $\eta_2 \sim \epsilon$ if $u_3^* \neq 0$. On the other hand it is clear that $\eta_2 = 0$ and since this is going to be the case throughout this paragraph we write

$$\eta = \eta_2$$

In view of II1.3.3 - 5 we can now write

$$\begin{aligned} \text{a. } m_2' &= s^{2-\eta} \left(m_2 + 4(N+1)u_4 g_{00}^{01} \right. \\ &\quad + 4u_3^2 g_{2k,0}^{11} - 4(N+1)m_2 u_4 g_{00}^{02} \\ &\quad \left. - 4u_3^2 m_2 g_{2k,0}^{12} \right) \end{aligned}$$

II1.3.6

$$\begin{aligned} \text{b. } u_3' &= s^{\epsilon/2-\eta} \left(u_3 - 4(N+1)u_3 u_4 g_{00}^{02} \right. \\ &\quad \left. - 8u_3^3 g_{2k,0}^{12} \right) \end{aligned}$$

$$u_4' = s^{\epsilon-2\eta} \left(u_4 - 4(N+7)u_4^2 g_{00}^{02} \right.$$

$$\text{c. } \left. - 3 \cdot 2^3 u_3^2 u_4 g_{2k,0}^{12} - 4u_3^4 g_{4k,0}^{21} \right)$$

Since in the equations b. and c. above there is no m , we can find u_k by linearising a.

We find

$$\Delta' m_2 = s^2 - \frac{8}{3} u_3^{*2} A_{2k,0}^{12} - 4(N+1) u_4^* A_{00}^{02} \Delta m_2$$

and so

$$\begin{aligned} \text{II1.3.7 } \frac{1}{v} &= 2 - \frac{8}{3} u_3^{*2} A_{2k,0}^{12} - 4(N+1) A_{00}^{02} u_4^* + \dots \\ &= 2 - \frac{2}{3} A_{00}^{02} \left(u_3^{2(1+t)} + 6(N+1) u_4^* \right) + \dots \end{aligned}$$

The two first sets of solutions of III.3.6 are

$$\begin{array}{lcl} \text{S1)} & u_3^* = 0 & \rightarrow \eta = 0 \\ & u_4^* = 0 & \nu = \frac{1}{2} \end{array}$$

which is the "anisotropic Gaussian" solution (AhFi), (Ah).

$$\begin{array}{lcl} \text{S2)} & u_3^* = 0 & \eta = 0 \\ & u_4^* = \frac{\epsilon}{A_{00}^2} \frac{1}{4(N+7)} & \rightarrow \nu = \frac{1}{2} + \frac{\epsilon(N+1)}{4(N+7)} \end{array}$$

which is the "anisotropic Heisenberg" solution (Ah).

Of course there is no surprise here since for $u_3^* = 0$ the model splits into a Heisenberg model and a Gaussian one.

The main interest of our model lies of course in the possibility that $u_3^* \neq 0$

Following appendix A4 we know that

$$A_{00}^2 = u_4 (1+t)^{-3/2} \quad \text{and} \quad g_{4h,0}^{22} = \frac{1}{8} K_4 (1+t)^{-5/2} \log s$$

We define then

$$x^2 = u_3^2 (1+t)^{-1}$$

III.3.8

$$y \equiv u_4$$

$$A_{00}^2 \equiv A$$

Then the equations III.3.6 a. and b. become, for $u_3^* \neq 0$,

$$\text{a.} \quad \frac{5}{3} A x^2 + 4(N+1) A y - \frac{\epsilon}{2} = 0$$

III.3.9

$$\text{b.} \quad \frac{1}{2} A x^4 + \frac{16}{3} A x^2 y - \epsilon y + 4(N+7) A y^2 = 0$$

These equations lead to

$$S3) \quad x^2 = -\frac{3\epsilon}{A} \frac{16N-44 + 6(N+1)(2N-9)^{\frac{1}{2}}}{L(N)}$$

$$y = \frac{\epsilon}{2A} \frac{18N+3 + 15(2N-9)^{\frac{1}{2}}}{L(N)}$$

with

$$L(N) = 72N^2 - 76N + 452 > 0 \quad \forall N$$

Since we want y to be real we have to ask that $N \geq 4,5$ that is $N-1 \geq 3,5$

On the other hand there is no restriction stemming from x^2 . It should be negativ and this is the case for every $N-1 \geq 3,5$.

$$S4) \quad x^2 = -\frac{3\epsilon}{A} \frac{16N-44 - 6(N+1)(2N-9)^{\frac{1}{2}}}{L(N)}$$

$$y = \frac{\epsilon}{2A} \frac{18N+3 - 15(2N-9)^{\frac{1}{2}}}{L(N)}$$

and again $N-1 \geq 3,5$. As long as we are interrested in $x^2 < 0$ this solution is of no interrest for integer N . In fact for $N > 5$, $x^2 > 0$ and for $N = 5$ $x = 0$ recovering the Heisenberg solution for $N = 5$.

We insert S3) in III.3.5 and III.3.7 and we find

$$III.3.9 \quad \eta = \epsilon \frac{16N-44 + 6(N+1)(2N-9)^{\frac{1}{2}}}{L(N)}$$

$$\frac{1}{\nu} = 2 \left(1 - \frac{\epsilon}{L(N)} (18N^2 + 5N + 47 + 9(N+1)(2N-9)^{\frac{1}{2}}) \right)$$

$$\nu = \frac{1}{2} + \frac{\epsilon}{2L(N)} (18N^2 + 5N + 47 + 9(N+1)(2N-9)^{\frac{1}{2}})$$

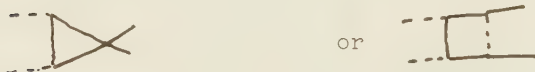
For $N \rightarrow \infty$ we don't find the spherical model result (St)

$$\nu = \frac{1}{d-2} \quad \text{but} \quad \nu = \frac{2}{d}$$

The surprise comes of course from $\eta \sim \epsilon$ and this justifies the title of this paragraph.

§ III.4 The case $w_4 \neq 0$, $u_4 = 0$

Before discussing this case let us see if we are justified in putting $u_4 = 0$. The first thing to notice is that $u_4 = 0$ is a solution of III.2.3 that is, it corresponds to a possible fixed point. But we can say more. If we start with the free energy III.1.7 but with $u_4 = 0$, renormalisation doesn't introduce, at least in lowest order, such a term. In terms of diagrams u_4 is produced by



But both are proportional to q^2 and so they introduce irrelevant terms. So $u_4 = 0$ can be viewed either as a fixed point solution, in which case we should investigate if it introduces instabilities, or as corresponding to a simpler model.

From III.2.2 b. we see that if $u_4 = 0$ there is no change in the form of v as given by III.3.7 and of course there is no change in the form of η in III.3.5. In general, inspection of III.2.1-3 shows that the equations of u_2 , u_4 and m_2 are the same as in §III.3 and hence nothing is altered to the solutions. The remaining equations for m_1 , e_1 , w_4 are just the Ising model (WiKo) equations:

$$e_1' = J_1^2 s^{-d-2} e_1$$

$$m_1' = J_1^2 s^{-d} (m_1 + 12 w_4 g_{00}^{10} - 12 m_1 w_4 g_{00}^{20})$$

$$w_4' = J_1^4 s^{-3d} (w_4 - 36 w_4 g_{00}^{20})$$

with the fixed points (apart from $m_1^* = w_4^* = 0$)

$$w_4^* = \frac{\epsilon}{36 A_{0,0}^2}, \quad m_3^* = -12 w_4^* \frac{g'_{0,0}}{1-s^{-2}}$$

and

$$\eta_1 = 0 \quad \nu_1 = \frac{1}{12} + \frac{1}{12} \epsilon$$

We have again put $t_1 = t_2 = t$ the justification being the same as in §III.3.

So the third order coupling leaves unchanged the behaviour of the ϕ_1 field but has an important influence on $\{\phi_{n \neq 1}\}$.

It remains to investigate if fixed points with $w_4, \nu_4 \neq 0$ exist such that $y^4 \neq 0$ and negative. Unfortunately we are forced to discuss a fourth order equation with coefficients which are polynomials in N . It turns out that a full discussion implies the solution of an equation of order twelve, which can be done only with numerical means. Without going into details we mention that for $N = 5$ there exists one fixed point satisfying the above conditions with $Ax^4 = -0.002$. As far as the possibility of $\eta \sim \epsilon$ is concerned we see that this is again the case.

§III.5 The stability of the solutions.

To study the stability of the solutions we must linearise the equations III.2.1-3. But throughout this discussion we are going to take $w_4 = 0$ from the very beginning and we won't consider at all its contribution.

According to § III.3 the first class of solutions is characterised by $w_4 = 0$. This class was subdivided into the cases

1. $\gamma_1 = s^{d/2}$
2. $\gamma_1 = s^{1+d/2}$

Consider first the first case.

We saw in §III.3 that this choice of γ_1 was done to keep $m_1 = \text{constant}$ and it followed that $u_3^* = 0$. The model splits into two parts, a trivial one and a Heisenberg, characterised by the fixed points m_3^* and u_4^* . If we set

$$\mu(m_3, u_4)$$

we find (by linearisation of III.3.2 b. c.)

$$\delta\mu' = L_1 \delta\mu$$

and L_1 has the eigen value

$$\lambda_2 = s^\epsilon \quad \text{for } u_4^* = 0 \quad (m_3^* = 0)$$

$$\text{and } \lambda_2 = s^{-\epsilon} \quad \text{for } u_4^* = \frac{\epsilon}{4(N+7)A_{00}^2}$$

i.e. the usual Heisenberg behaviour.

As it is expected $u_3^* = 0 = w_4^*$ don't give rise to instabilities in this case. We find for u_3 , by linearising III.2.3 c.

$$\begin{aligned} \Delta' u_3 &= s^{-1+\epsilon} \Delta u_3 \\ \text{and } \Delta' u_3 &= s^{-1+\epsilon} \frac{(5-N)}{2(N+7)} \Delta u_3 + \dots \end{aligned} \quad \begin{array}{l} \text{for } u_4^* = 0 \\ \text{for } u_4^* = \frac{\epsilon}{4(N+7)A_{00}^2} \end{array}$$

On the other hand (III.2.3.a)

$$\Delta' w_4 = s^{-4+\epsilon} \Delta w_4$$

We conclude that this case is really of no interest.

Consider now the second case.

For $u_4 = 0$ and $J_1 = S^{1+d/2}$ we know that $m_1^* = 0$, a Gaussian solution. For this case we have the solutions S1, S2, S3, S4.

S1 is characterised by $u_3^* = 0 = u_4^* = m_2^*$

This is again a Gaussian situation in $\{ \phi_{n \neq 1} \}$
Of course

$$\Delta' u_4 = S^\epsilon \Delta u_4, \quad \Delta' w_4 = S^\epsilon \Delta w_4$$

and

$$\Delta' u_3 = S^{\epsilon/2} \Delta u_3$$

S2 is slightly more interesting. It is characterised by

$$u_3^* = 0, \quad u_4^* = \frac{\epsilon}{4(N+7)A_{00}^2}$$

The ϕ_1 part still undergoes its Gaussian transition with

$$w_4^* = 0 = m_1^* \quad \text{and} \quad \Delta' w_4 = S^\epsilon \Delta w_4$$

On the other hand we find by III.2.3

$$\text{III.5.1} \quad \Delta' u_3 = S^{\epsilon \frac{5-N}{2(N+7)}} \Delta u_3$$

$$\Delta' u_4 = S^{-\epsilon} \Delta u_4$$

The behaviour of Δu_4 is usual Heisenberg behaviour. But we note that $N < 5$ gives rise to a peculiar behaviour. We saw in III.3 that $u_3^* = 0$ is the only acceptable solution for $N < 5$, as long as we want u_3 to be imaginary. In this region III.5.1 indicates an instability in u_3 . This situation is similar to the one of § II. As long as the third order coupling is different from zero it inhibits any continuous transition.

For $N > 5$ the situation changes. In fact there is no more instability in this case and even if u_3 is chosen different from zero, but small enough, the system is attracted to $u_3^* = 0$ spontaneously undergoing a Heisenberg transition. For $N = 5$ u_3 is marginal and higher order considerations are needed to decide if it remains marginal or not.

The next case we consider is S3 characterised by

$$u_3^{*2}(1+t)^{-1} x^2 = \frac{-3\epsilon}{A_{00}^2} \frac{16N - 44 + 6(N+1)(2N-9)^{\frac{1}{2}}}{L(N)}$$

$$u_4^* = \gamma = \frac{\epsilon}{2A_{00}^2} \frac{18N + 3 + 15(2N-9)^{\frac{1}{2}}}{L(N)}$$

which we accept for $N \geq 4.5$.

We linearise III.3.6 b. and c. and find with $\mu = (u_3, u_4)$

$$\delta \mu = L_1 \delta \mu$$

$$L_1 = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}$$

where $L_{12} \neq L_{21}$ and we hope that there exist real eigenvalues of this matrix.

In fact

$$L_{11} = s^{5/2} (1 - 5A x^2 \log s - 4(N+1)Ay \log s)$$

$$L_{12} = -s^{5/2} 4(N+1)H_4 x \log s$$

$$L_{21} = -s^6 \left(\frac{3^2}{3} xy A (1+t)^{-1/2} \log s + 2x^3 A (1+t)^{-1/2} \right)$$

$$L_{22} = s^6 \left(1 - \frac{16}{3} A x^2 \log s - 8(N+7) y A \log s \right)$$

eigenvalues are given by

$$\lambda_1 = s^{e(M_1(N) + M_2(N))}; \quad \lambda_2 = s^{e(M_1(N) - M_2(N))}$$

with $\lambda_1 > 1$ and $\lambda_2 < 1$, $\forall N \geq 5$.

$M_1(N)$ and $M_2(N)$ are very complicated and we don't write them down explicitly. We give their values for $N = 5$ and $N \rightarrow \infty$

For $N = 5$ $\lambda_1 = s^{3/12 \cdot e}$; $\lambda_2 = s^{-e}$

For $N \rightarrow \infty$ $\lambda_1 = s^e$; $\lambda_2 = s^{-e}$

So $u_i^* \neq 0$ introduces a relevant variable and the fixed point is unstable.

II2 An equivalent model.

In this paragraph we want to discuss the following model

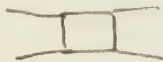
II2.1

$$\begin{aligned}
 F_1 = & \frac{1}{2} \int^{\wedge} (\rho_0 + \delta_0 q^2 + \sigma_0 (q^4)^2) (\vec{\Phi}(\vec{q}), \vec{\Phi}(-\vec{q})) \\
 & + \frac{1}{2} \int^{\wedge} (\rho_1 + \delta_1 q^2 + \sigma_1 (q^4)^2) \sum_{n \neq m}^{\vee} \phi_n(\vec{q}) \phi_m(-\vec{q}) \\
 & + \int^{\wedge} \kappa q_1^4 \sum_{n=1}^{\vee} \phi_n(\vec{q}) (\vec{\Phi}(\vec{q}_1), \vec{\Phi}(\vec{q}_2)) \delta(\sum_{i=1}^4 \vec{q}_i) \\
 & + \int^{\wedge} \lambda_1 (\vec{\Phi}(\vec{q}_1), \vec{\Phi}(\vec{q}_2)) (\vec{\Phi}(\vec{q}_3), \vec{\Phi}(\vec{q}_4)) \delta(\sum_{i=1}^4 \vec{q}_i) \\
 & + \int^{\wedge} \lambda_2 (\vec{\Phi}(\vec{q}_1), \vec{\Phi}(\vec{q}_2)) \sum_{n \neq m}^{\vee} \phi_n(\vec{q}_3) \phi_m(\vec{q}_4) \delta(\sum_{i=1}^4 \vec{q}_i) \\
 & + \int^{\wedge} \lambda_3 \sum_{\substack{n \neq m \\ \# \neq \# \\ k \neq l}}^{\vee} \phi_n(\vec{q}_1) \phi_m(\vec{q}_2) \phi_k(\vec{q}_3) \phi_l(\vec{q}_4) \delta(\sum_{i=1}^4 \vec{q}_i)
 \end{aligned}$$

where

$$(\vec{\Phi}(\vec{p}), \vec{\Phi}(\vec{q})) = \sum_{n=1}^{\vee} \phi_n(\vec{p}) \phi_n(\vec{q})$$

The problem in such a model is that direct calculations are intractable, the λ_2, λ_3 terms making the recursion relations very complicated. We cannot any more choose solutions $\lambda_2 = \lambda_3 = 0$ because through



κ is involved in both equations and $\lambda_2 = \lambda_3 = 0$ implies $\kappa = 0$ which leads to a trivial model. In order to obtain some information about II2.1 we will try to express the coupling constants in terms of the coupling constants of III.1.7. Let us then rewrite III.1.7 like

II2.2

$$\begin{aligned}
F = & \frac{1}{2} \int^{\wedge} (m_1' + e_1' q^2 + t_1' (q^4)^2) \phi_1(\vec{q}) \phi_1(-\vec{q}) \\
& + \frac{1}{2} \int^{\wedge} (m_2' + e_2' q^2 + t_2' (q^4)^2) (\vec{\phi}(\vec{q}), \vec{\phi}(-\vec{q})) \\
& + \int^{\wedge} u_3' q_1^4 \phi_1(\vec{q}_1) (\vec{\phi}(\vec{q}_2), \vec{\phi}(\vec{q}_3)) \delta(\sum_{i=1}^3 \vec{q}_i) \\
& + \int^{\wedge} w_4' \phi_1(\vec{q}_1) \phi_1(\vec{q}_2) \phi_1(\vec{q}_3) \phi_1(\vec{q}_4) \delta(\sum_{i=1}^4 \vec{q}_i) \\
& + \int^{\wedge} v_4' \phi_1(\vec{q}_1) \phi_1(\vec{q}_2) (\vec{\phi}(\vec{q}_3), \vec{\phi}(\vec{q}_4)) \delta(\sum_{i=1}^4 \vec{q}_i) \\
& + \int^{\wedge} u_4' (\vec{\phi}(\vec{q}_1), \vec{\phi}(\vec{q}_2)) (\vec{\phi}(\vec{q}_3), \vec{\phi}(\vec{q}_4)) \delta(\sum_{i=1}^4 \vec{q}_i)
\end{aligned}$$

II2.2 is identical to III1.1.7 if

$$m_1' = m_1 - m_2$$

$$m_2' = m_2$$

$$e_1' = e_1 - e_2$$

$$e_2' = e_2$$

$$u_3' = u_3$$

II2.3

$$w_4' = w_4 + u_4 - v_4$$

$$v_4' = v_4 - 2u_4$$

$$u_4' = u_4$$

$$t_1' = t_1 - t_2$$

$$t_2' = t_2$$

and because

$\int q_1^k \phi_1(\vec{q}_1) \phi_1(\vec{q}_2) \phi_1(\vec{q}_3) \delta(\sum_{i=1}^3 \vec{q}_i)$ is zero.

We now perform a rotation

$$\vec{\phi} \rightarrow R \vec{\phi}$$

such that

$$\phi_i = \frac{1}{\sqrt{N}} (\phi_i + \phi_i + \dots + \phi_i)$$

Then $F \rightarrow F\{\vec{\phi}\} = F\{R\vec{\phi}\}$

But of course $(\vec{\phi}, \vec{\phi}) = (R\vec{\phi}, R\vec{\phi})$

and so

$$\begin{aligned} F' = & \frac{1}{2} \int (m_1' + e_1' q^2 + t_1' (q^4)^2) \frac{1}{N} \sum_{n,m=1}^N \phi_n(\vec{q}) \phi_m(-\vec{q}) \\ & + \frac{1}{2} \int (m_3' + e_3' q^2 + t_3' (q^4)^2) (\vec{\phi}(\vec{q}), \vec{\phi}(-\vec{q})) \\ & + \int \frac{u_3'}{\sqrt{N}} q_1^k \sum_{n=1}^N \phi_n(\vec{q}_1) (\vec{\phi}(\vec{q}_1), \vec{\phi}(\vec{q}_3)) \delta(\sum_{i=1}^3 \vec{q}_i) \\ & + \int \frac{u_4'}{N^2} \sum_{n,m,l=1}^N \phi_n(\vec{q}_1) \phi_m(\vec{q}_2) \phi_l(\vec{q}_3) \phi_l(\vec{q}_4) \delta(\sum_{i=1}^4 \vec{q}_i) \\ & + \int \frac{v_4'}{N} \sum_{n,m=1}^N \phi_n(\vec{q}_1) \phi_m(\vec{q}_2) (\vec{\phi}(\vec{q}_3), \vec{\phi}(\vec{q}_4)) \delta(\sum_{i=1}^4 \vec{q}_i) \\ & + \int u_4' (\vec{\phi}(\vec{q}_1), \vec{\phi}(\vec{q}_3)) (\vec{\phi}(\vec{q}_3), \vec{\phi}(\vec{q}_4)) \delta(\sum_{i=1}^4 \vec{q}_i) \end{aligned}$$

=

$$\begin{aligned}
&= \frac{1}{2} \int^{\wedge} (\rho_0 + \delta_0 q^2 + \sigma_0 (q^4)^2) (\vec{\Phi}(\vec{q}), \Phi(-\vec{q})) \\
&+ \frac{1}{2} \int^{\wedge} (\rho_1 + \delta_1 q^2 + \sigma_1 (q^4)^2) \sum_{n \neq m}^{\infty} \Phi_n(\vec{q}) \Phi_m(-\vec{q}) \\
&+ \int^{\wedge} \kappa q_1^4 \sum_{n=1}^{\infty} \Phi_n(\vec{q}_1) (\vec{\Phi}(\vec{q}_2), \vec{\Phi}(\vec{q}_3)) \delta(\sum_{i=1}^3 \vec{q}_i) \\
&+ \int^{\wedge} \lambda_1 (\vec{\Phi}(\vec{q}_1), \vec{\Phi}(\vec{q}_2)) (\vec{\Phi}(\vec{q}_3), \vec{\Phi}(\vec{q}_4)) \delta(\sum_{i=1}^4 \vec{q}_i) \\
&+ \int^{\wedge} \lambda_2 (\vec{\Phi}(\vec{q}_1), \vec{\Phi}(\vec{q}_2)) \sum_{n \neq m} \Phi_n(\vec{q}_3) \Phi_m(\vec{q}_4) \delta(\sum_{i=1}^4 \vec{q}_i) \\
&+ \int^{\wedge} \lambda_3 \sum_{\substack{n \neq m \\ n, m \neq 0}} \Phi_n(\vec{q}_1) \Phi_m(\vec{q}_2) \Phi_k(\vec{q}_3) \Phi_l(\vec{q}_4) \delta(\sum_{i=1}^4 \vec{q}_i)
\end{aligned}$$

which is just II2.1 with

$$\rho_0 = \frac{m_1'}{N} + m_2' ; \quad \delta_0 = \frac{e_1'}{N} + e_2$$

$$\rho_1 = \frac{m_1'}{N} ; \quad \delta_1 = \frac{e_1'}{N}$$

$$\kappa = \frac{u_3'}{\sqrt{N}} ; \quad \sigma_0 = \frac{t_1'}{N} + t_2'$$

$$\lambda_1 = u_4' + \frac{v_4' + 3w_4'}{N^2} ; \quad \sigma_1 = \frac{t_1'}{N}$$

$$\lambda_2 = \frac{v_4'}{N} + 6 \frac{w_4'}{N^2}$$

$$\lambda_3 = \frac{w_4'}{N^2}$$

Or with II2.5

$$\rho_0 = \frac{m_1 - m_2}{N} + m_2 ; \quad \rho_1 = \frac{m_1 - m_2}{N}$$

$$\delta_0 = \frac{e_1 - e_2}{N} + e_2 ; \quad \delta_1 = \frac{e_1 - e_2}{N}$$

$$\sigma_0 = \frac{t_1 - t_2}{N} + t_2 ; \quad \sigma_1 = \frac{t_1 - t_2}{N}$$

$$\kappa = \frac{u_3}{\sqrt{N}}$$

$$\lambda_1 = u_4 + \frac{v_4 - 2u_4}{N} + 3 \frac{w_4 + u_4 - v_4}{N^2} ;$$

$$\lambda_2 = \frac{u_4 - v_4}{N} + 6 \frac{w_4 + u_4 - v_4}{N^2} ; \quad \lambda_3 = \frac{w_4 + u_4 - v_4}{N^2}$$

By inverting these relations we find

$$\begin{aligned} m_1 &= \rho_0 + \rho_1 (1-N) & e_1 &= \delta_0 + \delta_1 (1-N) & t_1 &= \sigma_0 + \sigma_1 (1-N) \\ m_2 &= \rho_0 - \rho_1 & e_2 &= \delta_0 - \delta_1 & t_2 &= \sigma_0 - \sigma_1 \end{aligned}$$

II2.5

$$u_3 = \sqrt{N} u$$

$$w_4 = \lambda_3 (N^2 - 6N + 3) + \lambda_2 (N-1) + \lambda_1 ;$$

$$v_4 = -6\lambda_3 (N-1) + \lambda_2 (N-2) + 2\lambda_1 ; \quad u_4 = 3\lambda_3 - \lambda_2 + \lambda_1$$

So the situation is the following :

Given II2.1 we may through II2.2-5 find III.1.7. In terms of the results of § III we can now discuss II2.1.

In fact we have :

a. For m_1 constant (i.e. $J_1 = \delta^{d/2}$) and $e_2 = 1$ we know that $e_1 \rightarrow 0, w_4 \rightarrow 0, v_4 \rightarrow 0, u_3 \rightarrow 0$. Inserting in II2.4 or II2.5 we find

$$u^* = 0 ; \quad \lambda_1^* = u_4^* - 2 \frac{u_4^*}{N} + 3 \frac{u_4^*}{N^2}$$

$$\lambda_2^* = - 2 \frac{u_4^*}{N} + \frac{6 u_4^*}{N^2} ; \quad \lambda_3 = \frac{u_4^*}{N^2}$$

If we calculate the exponents to lowest order in $1/N$ we may set $\lambda_2^* = \lambda_3^* = 0$. This is consistent with $\delta_0 = 1 - 1/N$ if $e_1 = 0$ and $e_2 = 1$. So we have again a Heisenberg model.

b. $m_1 = 0 = w_4 = v_4$ leads to a $u^* = u_4^* / \sqrt{N} \neq 0$ and hence to an $\eta \sim t$ if $N > 5$ as we have seen. Again we drop λ_2 and λ_3 . But in this case we set $e_1 = e_2 = 1$ and $t_1 = t_2 = t$ which permits to neglect δ_1 and σ_1 .

c. $m_1 = 0 = v_4, w_4 \neq 0$

Again we don't learn much new if we do the same approximation.

The interesting point is the condition $t_1 = t_2 = t$. Strictly speaking this leads by II2.4 to $\sigma_1 = 0, \sigma_0 = t$. But the third order term certainly contributes to σ_1 . On the other hand we know that t doesn't influence the critical exponents. The meaning of $\sigma_1 = 0$ is to be taken in the sense that if we consider σ_1 as an expansion parameter we may drop it in lowest order since $\sigma_1 = \mathcal{O}(1/N)$. The same is true for δ_1 . So we have the possibility of fixing δ to a constant by an appropriate choice of but this constant is going to be $\sim 1/N$.

II3 Comments on third order couplings with higher order wave-vector dependence.

We have studied, with the help of examples, two types of third order couplings i.e. constant and linear in the wave-vector. We will in this paragraph try to show that a higher order wave-vector dependence has no influence in the behaviour of the system. A model in this sense has been calculated in (MaSz).

The argument is of the same type as in chapter I except that here the coupling constant will be irrelevant, that is, equal to zero in the scaling limit and introducing no instability.

There is not much reason to invent a model with any kind of restrictions and so the discussion will be more qualitative than before.

We will choose a general third order coupling of the type

$$\int q^i q^j u_{ijk} \phi_i \phi_j \phi_k$$

Unlike the linear situation in chapter II the third order coupling of this kind produces tadpoles.

Example :

$$\int (q_i^i)^2 u_i \phi(\vec{q}_1) \phi(\vec{q}_2) \phi(\vec{q}_3) \delta(\sum_{i=1}^3 \vec{q}_i)$$

leads to a diagram

$$\text{tadpole} = 2 u_i \int_{\Lambda/2}^{\Lambda} (p^i)^2 G_0(\vec{p}) \frac{d\vec{p}}{(2\pi)^d}$$

which is of course different from zero.

It follows that some procedure in the spirit of chapter I must be applied to eliminate such terms.

Assume that we do that.

Then we must define some constant u such that the transformation

$$\phi \rightarrow \psi + u$$

eliminates the linear term. But then a third order term with constant coupling constant is introduced by the ϕ^3 term.

It follows that we are led to include in the model such a term.

Before considering such a case let us illustrate the irrelevance of a third order coupling of the type

$$\sum_{\substack{\alpha, \beta, \gamma=1 \\ \alpha \neq \beta \neq \gamma}}^n \int u_3(\vec{q}_1, \vec{q}_2) \phi_\alpha(\vec{q}_1) \phi_\beta(\vec{q}_2) \phi_\gamma(\vec{q}_3) \delta(\sum_{i=1}^3 \vec{q}_i)$$

with $u_3(\vec{q}_1, \vec{q}_2)$ quadratic in the wave-vectors and chosen such that $\langle \phi_\alpha^2 \rangle = \langle \phi_\beta^2 \rangle \neq 0$ and $\langle \phi_\gamma \rangle = 0$

The arguments follows then the lines of the first chapter. The recursion equation for u_3 would be of the type

$$u_3 = s^{-1 + \epsilon_2} (u_3 + f(u_3, u_4))$$

and for exactly the same reasons we see that $u_3^* = 0$ must be equal zero.

On the other hand diagrams like



don't diverge for $s \rightarrow \infty$

but they tend to be constants. Any way in the limit $s \rightarrow \infty$ and as far as the approach to the critical point is concerned only the diagram's linear in u_3 can remain which are at best logarithmically divergent in s . So, in lowest order in ϵ for example, the linearised equation for u_3 would look like

$$\Delta' u_3 = s^{-1 + \epsilon_2} + g(u_4) \Delta u_3$$

with $g(u_4) \sim \epsilon$ at least

So u_3 doesn't cause any instability and $u_3^* = 0$ so u_3 is irrelevant that is, we can forget it.

But although u_3 is irrelevant it generates through a constant third order coupling of a form corresponding to the conditions $\langle \phi_\alpha^2 \rangle = \langle \phi_\beta^2 \rangle$ and $\langle \phi_\gamma \rangle = 0$, and this, we know, is a coupling with a fixed point zero, but instable. So the conclusion is that it is not sensible to just introduce a u_3 which is quadratic in the wave-vector but rather a coupling

$$u_3(\{\vec{q}_i\}) = u_3 + u_3^3(\{\vec{q}_i\})$$

and just discard the second term as being irrelevant.

The situation is not changed by the relaxation of the condition $\langle \phi_i \rangle = 0$. We know that we need a linear transformation

$$\phi \rightarrow \gamma + u$$

to eliminate the tadpole.

But this transformation leaves the wave-vector dependent coupling unchanged and so again through the recursion relation

$$u_3 \sim g^{-1+\epsilon/2} (u_3 + \dots)$$

it follows that $u_3^* = 0$ and leads to no instability.

The situation is not changed even if we relax $\langle \phi_i^2 \rangle = \langle \phi_i^* \phi_i \rangle$. Actually in the equation for u_3 i.e.

$$u_3 \sim g^3 s^{-d-2} (u_3 + \dots)$$

we can choose g such that there exists a solution $u_3^* \neq 0$ and this in all cases mentioned above, but careful consideration of the corresponding recursion equations shows that we cannot calculate any more in orders of ϵ and so the perturbative approach to the R.G. loses its meaning.

It becomes clear that if we consider higher $\bar{q}$ -dependences they lead to "even more" irrelevant third order coupling parameters.

A1. The Ising model as an illustration of §05..

In this appendix we follow again (Ma) and (WiKo). We will consider the continuous version of the Ising model given by

$$\begin{aligned}
 F &= \frac{1}{2} \int^{\Lambda} G_0^{-1}(\vec{q}) \phi(\vec{q}) \phi(-\vec{q}) + u_4 \int^{\Lambda} \phi(\vec{q}_1) \phi(\vec{q}_2) \phi(\vec{q}_3) \phi(\vec{q}_4) \delta(\sum_{i=1}^4 \vec{q}_i) \\
 &= F_0 + F_I \\
 &= F_0 + X
 \end{aligned}$$

with $G_0^{-1}(\vec{p}) = (m + c p^2)^{-1}$

The calculation is going to serve to introduce the diagrammatic rules. We are going to find ν and η in terms of m^2 , u_4^2 in lowest order in ϵ and then relent. We want to calculate (see 05.19).

$$F_0 = - \langle \exp(-F_I) \rangle = -1 \sum_{\epsilon_i},$$

with the assumption $m \sim \epsilon$ and $u_4 \sim \epsilon$ we write

$$\langle \exp(-F_I) \rangle = -1 \sum_{\epsilon_i} = \langle -X + \frac{1}{2} X X + \dots \sum_{\epsilon_i},$$

$$= - \text{loop} + \frac{1}{2} \text{bubble} + \frac{1}{2} \text{triangle} + \dots$$

$$\begin{aligned}
 &= -6 u_4 \int^{\Lambda_s} \phi(\vec{q}) \phi(-\vec{q}) \int^{\Lambda} \frac{d\vec{p}}{(2\pi)^d} G_0(\vec{p}) + A \epsilon^2 \\
 &+ 36 u_4^2 \int^{\Lambda_s} \phi(\vec{q}_1) \phi(\vec{q}_2) \phi(\vec{q}_3) \phi(\vec{q}_4) \delta(\sum_{i=1}^4 \vec{q}_i) \\
 &\quad + \int^{\Lambda} G_0^2(\vec{p}) \frac{d\vec{p}}{(2\pi)^d} \\
 &\quad + \mathcal{O}(q^2 \epsilon^2) + \mathcal{O}(\epsilon^2)
 \end{aligned}$$

$A \epsilon^2$ comes from  $|m=q=0$

and $\mathcal{O}(q^2 \epsilon^2)$ come from  and 

expanded in q^2

We have dropped all the vacuum graphs like 

and we don't consider one particle reducible ones although  needs special consideration (WiKo). In the next appendices we give only the values of the amputated graphs.

The rules for the diagrams are the following :

- 1) The momenta are labelled in the incoming sense.
- 2) Internal lines have momenta restricted to the shell $\Lambda_{1/2} < q < \Lambda$, external have momenta in $q < \Lambda_{1/2}$.
- 3) To each internal line associate a propagator

$$G_1(\vec{q}_i) \delta(\vec{q}_i + \vec{q}_2) (2\pi)^d$$

- 4) To each four point vertex associate a factor

$$u_4 \delta(\vec{q}_1 + \vec{q}_2 + \vec{q}_3 + \vec{q}_4) (2\pi)^d$$

- 5) Integrate internal and external lines.

Then consider $G_0(\vec{p})$

$$G_0(\vec{p}) = \frac{1}{e\rho^2} - m \frac{1}{(e\rho^2)^2} + \mathcal{O}(\epsilon^2)$$

We can then write

$$\begin{aligned} \langle \exp(-F_I) - 1 \rangle_{0,2}^{\epsilon} &= -6 u_4 \int^{\Lambda_{1/2}} \phi(\vec{q}) \phi(-\vec{q}) \int_{\Lambda_{1/2}}^{\Lambda} \frac{d\vec{p}}{(2\pi)^d} \frac{1}{e\rho^2} + A \epsilon^2 \\ &\quad + \mathcal{O}(q^2 \epsilon^2) \\ &\quad - 6 m u_4 \int^{\Lambda_{1/2}} \phi(\vec{q}) \phi(-\vec{q}) \int_{\Lambda_{1/2}}^{\Lambda} \frac{d\vec{p}}{(2\pi)^d} \frac{1}{e^2 \rho^4} \\ &\quad + 36 u_4^2 \int^{\Lambda_{1/2}} \phi(\vec{q}_1) \phi(\vec{q}_2) \phi(\vec{q}_3) \phi(\vec{q}_4) \delta\left(\sum_{i=1}^4 \vec{q}_i\right) \\ &\quad + \int_{\Lambda_{1/2}}^{\Lambda} \frac{d\vec{p}}{(2\pi)^d} \frac{1}{e^2 \rho^4} + \mathcal{O}(\epsilon^2) + \mathcal{O}(q^2 \epsilon^2) \end{aligned}$$

Returning to 05.19 and 05.21 we can write by introducing

$$\phi_2 = \mathcal{I} \phi'(\vec{q}) \quad \text{with} \quad \vec{q}' = s \vec{q}, \quad \vec{q} < \Lambda_{1/2}$$

$$G_0^{-1}(\vec{q}) = \mathcal{I}^2 s^{-d} (G_0^{-1}(\vec{q}_{1/2}) + \Sigma_s(\vec{q}_{1/2}))$$

where s^{-d} comes from the integration by the variable change $\vec{q} \rightarrow s \vec{q}$

So we have identified (see 05.10)

$$J_1 = J s^{-d/2} = s^{1-\eta/2}$$

and hence

$$J = s^{1+d/2-\eta/2}$$

To calculate η we need Σ_1 , see 05.24. But the first $\vec{q}$ -dependent diagram to contribute in Σ_1 is  which is of order ϵ^2 . So $\eta = 0 + \mathcal{O}(\epsilon^2)$ and $J_1 = J$, $J = s^{1+d/2}$

Let us now calculate ν . Following §05 and the previous calculations we can write ($\epsilon=1$, see page 016).

$$m' = s^2 \left(m + 12 u_4 \int_{N_3}^{\Lambda} \frac{d\vec{p}}{(2\pi)^d} \frac{1}{p^2} - 12 m u_4 \int_{N_1}^{\Lambda} \frac{d\vec{p}}{(2\pi)^d} \frac{1}{p^4} + \mathcal{O}(\epsilon^2) \right)$$

$$u_4' = s^{\epsilon} \left(u_4 - 36 u_4^2 \int_{N_1}^{\Lambda} \frac{d\vec{p}}{(2\pi)^d} \frac{1}{p^4} + \mathcal{O}(\epsilon^2) \right)$$

where we have expanded the integrals in powers of ϵ keeping the zero'th order term (see A2).

We can solve these equations and find m^* and u_4^* . Then, linearisation yields

$$\delta\mu' = L_1 \delta\mu$$

and diagonalising L_1 we find (see 05.11-14)

$$\lambda_1 = s^{2-12 u_4^* u_4}$$

and hence

$$\frac{1}{\nu} = 2 - 12 u_4^* u_4 \rightarrow \nu = \frac{1}{2} + 3 u_4^* u_4 + \mathcal{O}(\epsilon^2)$$

A2. The d-dimensional integral.

The space dimensions in different calculations in the work are fixed at $d = 4 - \epsilon$. This, of course, complicates the calculations. The idea is to expand the integrals around $\epsilon = 0$, since we calculate fixed points and critical exponents in powers of ϵ .

Consider then some d-dimensional integral

$$A2.1 \quad \int_a^b \frac{d\vec{p}}{(2\pi)^d} f(\vec{p}) = \left[\int_a^b dp \, p^{d-1} \int_0^\pi \sin^{d-2} \vartheta_1 d\vartheta_1 \dots f(p, \vartheta_1, \dots) \right] \times \\ \times K_d \left[\int_0^\pi \sin^{d-2} \vartheta_1 d\vartheta_1 \dots \right]^{-1}$$

with

$$K_d^{-1} = 2^{d-1} \pi^{d/2}$$

We can now set $d = 4 - \epsilon$ and expand around $\epsilon = 0$.

Lets work out a simple example where $f(\vec{p}) = f(p)$
Then

$$A2.2 \quad \int_a^b f(p) \frac{d\vec{p}}{(2\pi)^d} = K_4 \int_a^b f(p) p^3 dp + \epsilon \left[\frac{3}{2} K_{4-\epsilon} \int_a^b p^3 f(p) \right. \\ \left. - K_4 \int_a^b dp \, p^3 \log p f(p) \right]_{\epsilon=0}$$

Take now $f(p) = (m + p^2)^{-1}$

with $m \sim \epsilon$. This is of course the propagator corresponding to 06.1 or 06.2 and the integral corresponds to the diagram

Then 

$$\int_{\Lambda_1}^{\Lambda_2} \frac{d\vec{p}}{(2\pi)^d} (m + p^2)^{-1} = \frac{1}{2^4 \pi^2} \Lambda^2 (1 - s^{-2}) \\ + \frac{\epsilon}{2^4 \pi^2} \left[\log(2\pi^{1/2}) \Lambda^2 (1 - s^{-2}) \right. \\ \left. - \Lambda^2 [\log \Lambda - 1/2 - s^{-2} (\log \Lambda_1 - 1/2)] \right] \\ - \frac{1}{2^3 \pi^2} m \log s + \mathcal{O}(\epsilon^2)$$

A3. The diagram  in the case of constant third order coupling.

This is the situation of chapter I where it is stated that this diagram has no contribution to η since it is wave-vector independent.

This is so because

$$A3.1 \quad \omega = \int_{N_s}^{\wedge} \frac{d\vec{p}}{(2\pi)^d} (m + p^2)^{-1} (m + (q + \vec{p})^2)^{-1/2}$$

the integral corresponding to  can be calculated as

$$\omega = N_s \int_{N_s}^{\wedge} dp \, p \left[\int_0^{\wedge} \frac{\sin^2 \vartheta \, d\vartheta}{p^2 + q^2 + 2pq \cos \vartheta} \right] \left[\int_0^{\wedge} \sin^2 \vartheta \, d\vartheta \right]^{-1}$$

$$\text{But } \int_0^{\wedge} \frac{\sin^2 \vartheta}{p^2 + q^2 + 2pq \cos \vartheta} d\vartheta \approx \min(p^{-1}, q^{-1}); \quad \text{see (GrRy).}$$

and since $q < N_s$ and $p > N_s$ it follows from $p^{-1} < q^{-1}$ that ω is independent of q , the external wave-vector.

A4. The diagrams of §III.

$$\bigcirc = 2(N+1) g_{00}^{01} u_4 \delta_{N,1} - 2(N+1) g_{00}^{02} m_2 u_4 \delta_{N,1}$$

$$\text{---}\bigcirc\text{---} = g_{00}^{01} v_4 \delta_{N,1} - g_{00}^{02} m_2 v_4 \delta_{N,1}$$

$$\underline{\bigcirc} = g_{00}^{10} v_4 \delta_{N,1} - g_{00}^{20} m_1 v_4 \delta_{N,1}$$

$$\text{---}\bigcirc\text{---} = 6 g_{00}^{10} w_4 - 6 g_{00}^{20} m_1 w_4$$



this diagram is going to be calculated in detail in appendix A5.

$$\text{---}\bigcirc\text{---} = 4(N+1) q_1^k u_3 u_4 g_{00}^{02} \delta_{N,1}$$

$$\text{---}\bigcirc\text{---} = 2 q_1^k u_3 v_4 (g_{2k,0}^{12} (e_2 + t_2) + \text{sym.}) \delta_{N,1}$$

$$\text{---}\bigcirc\text{---} = -48 q_1^k u_3^3 g_{2k,0}^{12} \delta_{N,1}$$

$$\bigcirc\bigcirc = 8(N+7) g_{00}^{02} u_4^2 \delta_{N,1}$$

$$\text{---}\bigcirc\text{---} = 2 g_{00}^{20} v_4^2 \delta_{N,1}$$

$$\text{Diagram 1} = 4(N+1) g_{00}^{02} u_4 \bar{u}_4 \delta_{N,1}$$

$$\text{Diagram 2} = 12 v_4 w_4 g_{00}^{20} \delta_{N,1}$$

$$\text{Diagram 3} = 16 g_{00}^{11} v_4^2 \delta_{N,1}$$

$$\text{Diagram 4} = 72 g_{00}^{20} w_4^2$$

$$\text{Diagram 5} = -48 g_{24,0}^{12} u_3^2 u_4 \delta_{N,1}$$

$$\text{Diagram 6} = 2 g_{00}^{02} u_4^2$$

$$\text{Diagram 7} = -8 g_{24,0}^{21} u_3^2 v_4 \delta_{N,1}$$

$$\text{Diagram 8} = -8 g_{24,0}^{12} u_3^2 v_4 \delta_{N,1}$$

$$\text{Diagram 9} = -48 g_{24,0}^{21} u_3^2 w_4 \delta_{N,1}$$

$$\text{Diagram 10} = 96 u_3^4 g_{44,0}^{22} \delta_{N,1}$$

where as in § III

$\delta_{N,1} = 1 - \delta_{N,1}$, δ being the Kronecker symbol

$$g_{m_i, n_j}^{st} = \int_{N_j}^1 \frac{d\vec{p}}{(2\pi)^d} (p^i)^m (p^j)^n (G_{01}(\vec{p}))^s (G_{02}(\vec{p}))^t \Big|_{m_i=0}$$

and

$$G_{01}(\vec{p}) = (m_1 + e_1 p^2 + t_1 (p^4)^2)^{-1}$$

$$G_{02}(\vec{p}) = (m_2 + e_2 p^2 + t_2 (p^4)^2)^{-1}$$

$$(e_2 + t_2) g_{0,0}^{st} + \text{sym} = g_{0,0}^{st} (e_2 + t_2) + g_{0,0}^{ts} (e_1 + t_1)$$

We may add that we don't consider one-particle reducible graphs, that is graphs of the type :



This is so because they are either zero like



or because they are constants in respect to S , which means that their contribution is zero in the limit $S \rightarrow \infty$.

In our specific case these diagrams are



Along with the mentioned reasons for neglecting them, we notice that they give contributions to $q^2 u_4$ or $q^2 u_5$ which are both irrelevant.

The case $q^2 u_4$ is discussed in detail in (WiKo). As for $q^2 u_5$ we notice its order is $\epsilon^{3/2}$ and since it goes linearly in u_3 by  it should be considered.

Luckily we don't have to take it into account the reasons being essentially the same with the reasons given to discard u_6 in the Ising model (WiKo).

A5. The diagram

This is "the most important" diagram in the sense that through this diagram we find the contribution to η . Following the rules of A1 we find

$$\begin{aligned}
 & \text{---} \text{---} \text{---} \left(= 4! \frac{2}{3} \frac{1}{6} \frac{4}{5} \right) \\
 & = 4! \sum_{n, m \neq 1}^N \delta_{n, m} u_3^2 \int_{\Lambda_S}^{\Lambda} \frac{d\vec{q}_1}{(2n)^d} \frac{d\vec{q}_3}{(2n)^d} \frac{d\vec{q}_4}{(2n)^d} \frac{d\vec{q}_6}{(2n)^d} \\
 & \quad q_1^4 q_4^4 G_{0,1}(\vec{q}_1) \delta(\vec{q}_1 + \vec{q}_4) (2n)^d \\
 & \quad G_{0,2}(\vec{q}_3) \delta(\vec{q}_3 + \vec{q}_6) (2n)^d \\
 & \quad \delta(\vec{q}_1 + \vec{q}_2 + \vec{q}_3) (2n)^d \\
 & \quad \delta(\vec{q}_4 + \vec{q}_5 + \vec{q}_6) (2n)^d
 \end{aligned}$$

and

$$q_2, q_5 < \Lambda_S$$

Since we need this diagram up to first order in m_1 and m_2 we may expand $G_{0,1}(\vec{q}_1)$ and $G_{0,2}(\vec{q}_3)$ as

$$\begin{aligned}
 G_{0,1}(\vec{q}_1) &= (e, q_1^2 + t, (q_1^4)^2)^{-1} - m_1 (e, q_1^2 + t, (q_1^4)^2)^{-2} \\
 &= G_{0,1}^0(\vec{q}_1) - m_1 G_{0,1}^{\circ 1}(\vec{q}_1)^2 \\
 G_{0,2}(\vec{q}_3) &= G_{0,2}^0(\vec{q}_3) - m_2 G_{0,2}^{\circ 1}(\vec{q}_3)^2
 \end{aligned}$$

and hence

$$\begin{aligned}
 \text{---} \text{---} &= -4 \sum_{n \neq 1}^{\infty} u_3^2 \int_{\Lambda_3}^1 \frac{d\vec{q}_1}{(2\pi)^d} \frac{d\vec{q}_2}{(2\pi)^d} \cdot \\
 & (q_1^4)^2 \left(G_{0,1}^0(\vec{q}_1) G_{0,2}^0(\vec{q}_2) \right. \\
 & \left. - m_1 (G_{0,1}^0(\vec{q}_1))^2 G_{0,2}^0(\vec{q}_2) - m_2 G_{0,1}^0(\vec{q}_1) (G_{0,2}^0(\vec{q}_2))^2 \right) \\
 & \delta(\vec{q}_1 + \vec{q}_2 + \vec{q}_3) (2\pi)^d \\
 & \delta(-\vec{q}_1 + \vec{q}_2 - \vec{q}_3) (2\pi)^d + \mathcal{O}(\epsilon^3)
 \end{aligned}$$

We are going to consider explicitly the zeroth order term in m_i . The term proportional to m_1 is of no interest because as we have seen either m_1 is some constant and in this case $u_3^* = 0$ or m_1 is zero or $m_1 \sim \epsilon$ and so the contribution of this term to m_2 is $\sim \epsilon^2$. On the other hand the term proportional to m_2 is of interest since it is involved in ν in the case $u_3^* \neq 0$.

So we have to calculate

$$\rho = \int_{\Lambda_3}^1 \frac{d\vec{q}_1}{(2\pi)^d} \frac{d\vec{q}_2}{(2\pi)^d} (q_1^4)^2 G_{0,1}^0(\vec{q}_1) G_{0,2}^0(\vec{q}_2) \cdot \delta(\vec{q}_1 + \vec{q}_2 + \vec{q}_3) \delta(-\vec{q}_1 + \vec{q}_2 - \vec{q}_3) (2\pi)^{2d}$$

Actually we need it for $\ell_1 = \ell_2 = \ell$ and $t_1 = t_2 = t$, see III.1 and A6.

So

$$\rho = \int_{\Lambda_3}^1 \frac{d\vec{p}}{(2\pi)^d} (p^4)^2 G_{0,0}(\vec{p}) G_{0,0}(\vec{q}_r - \vec{p}) \delta(\vec{q}_r + \vec{q}_r - \vec{p}) (2\pi)^d$$

$$G_{0,0}(\vec{p}) = (p^2 + t(p^4)^2)^{-1}$$

and

We can now develop ρ in $\vec{q}_r$, since we are interested in the small q behaviour, to get the term contributing to $\Sigma(\vec{q})$:

$$\begin{aligned} \rho = & g_{2k,0}^{11} + (q_r^k)^2 (t g_{2k,0}^{12} + 4(1+t) g_{4k,0}^{13} \\ & + 4 g_{2k,2i}^{13} + \frac{1}{2} g_{00}^{11} - 2(1+t) g_{2k,0}^{12}) \\ & + q_r^2 (-g_{2k,0}^{12} + 4 g_{2k,2i}^{13}) + O(q^2) \end{aligned}$$

The first term contributes to m_2 and the last gives us

$$q^2 \frac{\partial \Sigma}{\partial q^2} / q=0$$

The factors t , $(1+t)$ appear because of the expansion in $\vec{q}_r$. We can now illustrate how the different g_i are calculated. Take for example

$$\begin{aligned} g_{2k,0}^{11} &= \int_{1/2}^1 \frac{d\vec{p}}{(\sqrt{\pi})^d} G_{00}^2(\vec{p}) (p^k)^2 \\ &= \int_0^\pi \int_{1/2}^1 d\rho \rho^{d-3} \frac{\cos^2 \vartheta \sin^{d-2} \vartheta}{(1+t \cos^2 \vartheta)^2} d\vartheta \\ &\quad \cdot K_d \cdot \left(\int_0^\pi \sin^{d-2} \vartheta d\vartheta \right)^{-1} \\ &\sim \int_0^\pi \frac{\cos^2 \vartheta \sin^{d-2} \vartheta}{(1+t \cos^2 \vartheta)^2} d\vartheta = \rho_1 \end{aligned}$$

Now for $|t| < 1$ we can expand the denominator

$$(1+t \cos^2 \vartheta)^{-2} = \sum_n \frac{(-t \cos^2 \vartheta)^n}{n!}$$

and hence (the series converging absolutely):

$$P_1 = \sum_m (m+1) (-t)^m \int_0^\pi \cos^{2m} \vartheta \sin^2 \vartheta d\vartheta$$

but

$$\frac{n}{2} \int_0^\pi \cos^n \vartheta \sin^m \vartheta d\vartheta = \frac{n}{2} \frac{\Gamma(\frac{n+1}{2}) \Gamma(\frac{m+1}{2})}{\Gamma(\frac{n+m+2}{2})}$$

so $P_1 = (1+t)^{-\frac{1}{2}}$
which gives

for $d = 4$

$$g_{2k,0}^{11} = \frac{1}{2} \Lambda^2 (1-s^{-2}) K_4 (1+t)^{-1/2}$$

The other g_i are calculated in a similar way.

The diagram just discussed is more complicated in the case $t_1 \neq t_2$. Although we don't discuss this case we note that this brings about the symmetrisation of the integrant mentioned in A4. The result is

$$\begin{aligned} \text{---} \bigcirc \text{---} &= -4 \sum_{n \neq i} u_3^2 \chi_{n,i} \left\{ g_{2k,0}^{11} - m_1 g_{2k,0}^{21} - m_2 g_{2k,0}^{12} + \right. \\ &+ (q^4)^2 \left[-\frac{1}{2} (t_1 g_{2k,0}^{21} + t_2 g_{2k,0}^{12}) + \right. \\ &+ 2(e_1 + t_1) g_{4k,0}^{31} + 2(e_2 + t_2) g_{4k,0}^{13} \\ &- 2 \left(e_1^2 g_{2k,2i}^{31} + e_2^2 g_{2k,2i}^{13} \right) + \frac{1}{2} g_{0,0}^{11} - 2(e_1 + t_1) g_{2k,0}^{21} \left. \right] \\ &+ q^2 \left[-\frac{1}{2} (e_1 g_{2k,0}^{21} + e_2 g_{2k,0}^{12}) \right. \\ &+ 2 \left(e_1^2 g_{2k,2i}^{31} + e_2^2 g_{2k,2i}^{13} \right) \left. \right] \\ &+ \delta(q^2) + \delta(\epsilon^3) \end{aligned}$$

A6. Some considerations on t_1 and t_2 .

We have mentioned in § III that putting $t_1 = t_2 = t$ is the easiest way to handle the model under consideration. We are going to try to justify this choice in this appendix.

Consider the equation for t_1 in the case $\gamma_1 = s^{1+\frac{1}{2}t_1}$. In the case $\gamma_1 = s^{\frac{1}{2}t_1}$, $t_1 \rightarrow 0$ but so does u_1 etc.

We find

$$t_1' = (t_1 - \frac{1}{2} \dots 0 \dots + \dots)$$

$$= t_1 + 2(N-1) u_1^2 g^{\frac{1}{2}t_1}$$

If we consider t_1 as an expansion parameter and we expand in lowest order we find

$$0 = 2(N-1) u_1^2 (1 - \frac{t_1}{2})$$

and hence either $u_1 = 0$ and the models decouple or $t_1 = 2$ which contradicts the expansion in t_1 . The corresponding equation for t_2 is more complicated but the result is the same.

At this stage we can try to change the definition of γ_1 to

$$\gamma_1 = s^{-(N-1) u_1^2 \frac{(1+t_1)^{\frac{1}{2}}}{t_1} u_4 + 1 + \frac{1}{2} t_1}$$

$$= s^{-(N-1) u_1^2 \frac{(1+t_1)^{\frac{1}{2}}}{t_1} u_4}$$

If $x^2 < 0$ as we wish, it turns out that $e_1 \rightarrow \infty$ and we have to change the model by putting $e_1 = 0$. If $x^2 > 0$ then $e_1 \rightarrow 0$. In any case t_1 becomes arbitrary and we can again put $t_1 = t_2 = t$. In this case it looks as if we introduce an η_1 by $\gamma_1 s^{-\eta_1} = s^{1-\eta_1}$ and so the critical exponents could depend on t . But if $t > 0$, u_4 turns out to be negative. Hence u_1 must be zero and the model becomes trivial again.

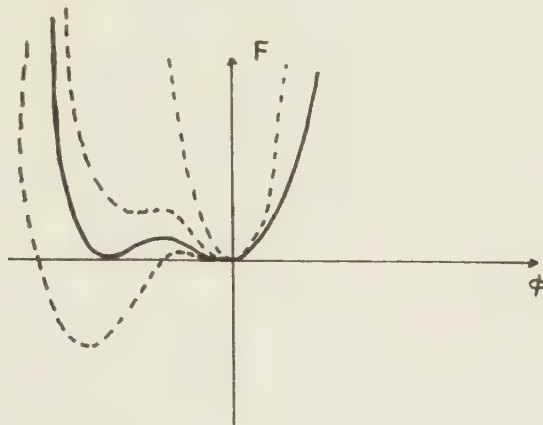
CONCLUSION

The original aim of this work was to give some classification of the behaviour of the models including third order couplings, in the scaling limit. This task turned out to be too hard and too long and the discussion is limited mostly to specific examples. The question is if and in what sense we have approached the original aim.

Studying the critical behaviour of a model means essentially two things. First to find out if the model undergoes a continuous or a discontinuous transition (or subclassifications of them). This is the qualitative part. Second to analyse quantitatively its behaviour i.e. to calculate the critical exponents, in the case of a continuous transition.

In general it was believed that Landau theory and renormalisation group theory gave rise to the same qualitative conclusions as far as continuity or discontinuity are concerned. It was shown by (AlAm) that this is not always the case and the counterexample is the liquid-gas transition model

If we plot the free energy versus ϕ we get the following picture :



It is clear from this picture that as ϕ changes continuously its value, let us say from t_c

a second minimum appears, discontinuously, at $\phi < 0$ and hence there is no continuous transition as long as

The renormalisation group approach contradicts this result showing the existence of a critical point (AlAm). This situation seems to be related to the (lack) of symmetry of the model. In fact we saw in §I1 that this qualitative difference disappears for models which have some rotational symmetry with no distinguished axis, or to be more specific where there is no mass splitting and tadpoles. There, as long as the constant part of the third order coupling parameter is different from nought, no continuous transition can take place.

On the other hand in the model of §I2 there is mass splitting and tadpoles. Here Landau theory permits a continuous transition for a suitable choice of the parameters.

But renormalisation group allows for a better understanding of the possible critical regions. In fact $w_3 = w_4 = 0$ is necessary in Landau theory too. But that they are irrelevant is not so evident. These are exactly the parameters that inhibit a continuous transition even in Landau theory and it is gratifying to see that they disappear "by themselves".

Of course the study of the above cases doesn't give a classification. It rather shows the variety of critical behaviours allowed by constant third order couplings even if the values of the critical exponents (when there are any) are the same as in the models with even terms only.

When we turn our attention to wave-vector dependent couplings the comparison with the "bulk" Landau theory stops. In fact derivative terms drop in this theory.

The model discussed is a non-symmetric one. Its critical behaviour is quite rich (there are many different fixed points). It is, quantitatively, a kind of surprise. In fact its critical exponents, corresponding to the fixed points with the third order coupling parameter different from zero, are different from the ones of models including only even couplings. In particular the anomalous dimension is proportional to $\epsilon = 4 - d$ and not to ϵ^2 . The surprise is that if we restrict the universality hypothesis to the propositions 1 and 2 on page 04 there should be no difference in the powers of ϵ . But we note that we can restate this hypothesis in terms of dimensions and symmetries. The two statements are not equivalent. For example the order parameters in the liquid-gas system and the Ising model have the same degrees of freedom, $N = 1$, but the models have not the same symmetry i.e. the liquid has no symmetry whatsoever and the Ising model has the reflection symmetry. Still, experimentally, the discussion of the value of the critical exponents for the liquid-gas transition is not closed yet. The model discussed in Ch.II has two relatives. First the model devised by (HaLuMa) to discuss superconductors.

Second the model introduced in (Sz1) to discuss perovskite structures. In this second case the third order term is of the form

$$\int dx \sum_{ij} (\partial_i \hat{\phi}_i(x)) \hat{\phi}_j(x) \hat{\phi}_j(x)$$

and hence $N = d$. This term has the full rotational symmetry. In first sight it would seem that $\eta \sim \epsilon$ since the diagram



introduces a term proportional to $g^2 u_3^2$. But what remains to be verified is if there exists a fixed point with $u_3^* \neq 0$. If this is the case both ways of stating the universality hypothesis mentioned before would be questionable and only proposition 3. on page 04 would be the way out. If this is not the case the second way of stating this hypothesis remains valid.

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